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Low-temperature thermodynamics of random-field Ising chains: exact results

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Abstract. We consider the random-field ferromagnetic Ising chain, for a class of continuous symmetric probability distributions of the random magnetic fields, of a diluted power-times-exponential type. This class of distributions is 'exactly solvable', in the sense that the disorder can be integrated explicitly, at any temperature. A detailed analysis of the low-temperature thermodynamics is presented. Exact expressions are obtained for the ground-state energy, the zero-temperature entropy, and the amplitude of the specific heat, which vanishes linearly at low temperature. The diluted symmetric binary distribution, where the magnetic fields can only assume the values $\pm H_B$ or zero, can be viewed as a limiting case of the class of exactly solvable distributions. The low-temperature physics of this discrete model is investigated in detail, including the exponential fall-off of the specific heat. These results are put in perspective with those of the 'exact solution', with emphasis on the crossover between various characteristic features of continuous and discrete field distributions, both at zero and finite temperature.

1. Introduction

This article reports on the continuation of our investigations of 'exactly solvable' classes of ferromagnetic Ising chains in the presence of quenched random magnetic fields [1–3]. Random-field Ising models owe most of their interest to the combined effects of frustration and randomness. Recent and comprehensive overviews of these matters can be found in [4, 5], which deal in particular with the roughness of domain walls, the lower critical dimension, and dynamical aspects.

The one-dimensional situation of random-field Ising chains is of physical interest, in spite of the absence of a phase transition at finite temperature. Indeed, the random-field Ising chain is a frustrated system, unlike the one-dimensional spin glass. As a consequence, there exist a large number of degenerate, or almost-degenerate, ground-states, so that the system exhibits a non-trivial low-temperature thermodynamics. Random-field Ising chains have been the subject of an abundant literature [6–13], which emphasizes the connection with products of 2×2 random matrices, and Lyapunov exponents. Although the problem may look simple, it turns out that only a limited amount of exact information is available, concerning either thermodynamical quantities or correlation functions, for special classes of distributions of the random fields. Let us mention in particular the following studies. The zero-temperature energy and entropy are known exactly for a general binary distribution [6]. The regime where the exchange coupling is much larger than the random fields has been investigated in [8]. The limit where the random fields are either zero or (plus or minus) infinity has been studied by an exact enumeration method [9], which we recall in the appendix.

In previous publications [1–3], we have presented several examples of 'exactly solvable' classes of random-field Ising chains. For these special distributions of the random magnetic fields—which are essentially exponential functions—we have shown how to evaluate both the free energy and the (connected) two-point correlation function, at any temperature. This approach, introduced by one of us, has been applied to a variety of linear and non-linear problems related to the physics of disordered one-dimensional systems [14–16]. Its essential step consists in an exact integration over the random variables of the problem. Let us emphasize that the method used in [1–3], which yields in particular exact expressions for the ground-state energy, zero-temperature entropy, and linear specific heat amplitude, is one of the very few approaches which permit a detailed analytical study of low-temperature thermodynamic properties.

In the present work, we aim at extending the analytical approach of [1–3] to a more general class of 'exactly solvable' random-field Ising chains, where the probability distribution of the random magnetic fields h_n is defined as follows. We set

$$h_n = H x_n. \tag{1.1}$$

The positive parameter H is a measure of the strength of disorder, and the x_n are dimensionless independent random variables, with the common symmetric power-times-exponential distribution

$$\tilde{R}(x) = \frac{p}{2}\frac{|x|^{\nu-1}\mathrm{e}^{-|x|}}{(\nu-1)!} + r\delta(x). \tag{1.2}$$

The parameter $\nu \geqslant 1$ is an arbitrary integer. The model is diluted, in the sense that only a fraction p of the spins experience a non-zero magnetic field. p can thus be referred to as the impurity concentration. The notation $r = 1 - p$ will be used throughout this work.

The class of 'exactly solvable' symmetric field distributions just defined contains as particular cases the situations studied previously. In the case $\nu = 1$ of a pure exponential distribution, we have studied the thermodynamical properties [1], with emphasis on the low-temperature regime, and the connected two-point correlation function [2]. In the case $\nu = 2$, the thermodynamical properties have been considered in [3].

One of the most interesting features of the distribution defined earlier is that it has a three-peak structure, for $\nu \geqslant 2$, since its continuous part has maxima for $x = \pm(\nu - 1)$, i.e. $h = \pm(\nu - 1)H$. When the integer ν is large, these maxima become sharper and sharper, so that we are left with the diluted symmetric binary distribution

$$R(h) = \frac{p}{2}\left[\delta(h - H_{\mathrm{B}}) + \delta(h + H_{\mathrm{B}})\right] + r\delta(h) \tag{1.3}$$

in the $\nu \to \infty$ limit, keeping the following product constant

$$H_{\mathrm{B}} = \nu H. \tag{1.4}$$

The parameter $H_{\rm B}$ will thus be referred to as the field strength of the binary model.

The random-field Ising model, with the power-times-exponential distribution (1.2), for arbitrary values of the integer ν, has already been studied within the mean-field approximation, and the replica method [17].

The present article reports on a detailed analytical study of the thermodynamical properties for the class of distributions defined earlier, by means of an exact solution of the problem, the parameter $\nu \geqslant 1$ being an arbitrary integer. We aim at putting a special emphasis on the low-temperature regime, and on the large-ν crossover behaviour to the binary limit (1.3). After having recalled some general formalism about the transfer-matrix method in section 2, we derive the exact solution of the model at finite temperature in section 3. The study of the low-temperature behaviour requires a more technical analysis, exposed in section 4, where we evaluate the ground-state energy, the zero-temperature entropy, and the amplitude of the specific heat, which is linear in temperature. Section 5 contains more explicit forms of our general results in several special cases of interest. The limiting case of the diluted binary distribution is investigated in section 6, as well as the large-ν crossover from a continuous to a discrete distribution, at zero temperature. In the concluding section 7, we discuss the extension of this crossover behaviour to finite but low temperatures. Finally, the appendix presents the solution of the $H \to \infty$ limit of the model, which has already been investigated in [9].

2. General formalism

The Hamiltonian of the ferromagnetic Ising chain in a random magnetic field reads as

$$\mathcal{H} = -J \sum_n \sigma_n \sigma_{n+1} - \sum_n h_n \sigma_n. \tag{2.1}$$

The exchange constant J is a fixed positive quantity, whereas the local fields h_n are independent random variables, with a common symmetric (even) probability distribution $R(h)\,\mathrm{d}h$.

According to the well-known transfer-matrix approach, the partition function Z_N at temperature $T = 1/\beta$ of a finite chain consisting of N sites, with periodic boundary conditions, reads

$$Z_N = \operatorname{tr} \prod_{n=1}^{N} T_n \qquad \text{with} \qquad T_n = \begin{pmatrix} \mathrm{e}^{\beta J+\beta h_n} & \mathrm{e}^{-\beta J+\beta h_n} \\ \mathrm{e}^{-\beta J-\beta h_n} & \mathrm{e}^{\beta J-\beta h_n} \end{pmatrix}. \tag{2.2}$$

As a consequence, the quenched free energy F per site is simply related to the Lyapunov exponent of the infinite matrix product, namely

$$-\beta F = \lim_{N\to\infty} \frac{1}{N} \ln \operatorname{tr} \prod_{n=1}^{N} T_n. \tag{2.3}$$

In order to evaluate this quantity, and following e.g. [1–3], we introduce a sequence of vectors (x_n, y_n) such that (x_n, y_n) is the image of (x_{n-1}, y_{n-1}) by the matrix T_n, and we consider the ratios (Riccati variables) $\rho_n = x_n/y_n$. These quantities obey the recursion formula

$$\rho_n = e^{2\beta h_n} \frac{e^{2\beta J} \rho_{n-1} + 1}{\rho_{n-1} + e^{2\beta J}}. \tag{2.4}$$

When the site label n becomes large, the distribution of the positive random variable ρ_n converges to a stationary limit distribution, which is invariant under the transform (2.4). The existence of this stationary distribution is essentially equivalent to Oseledec's theorem on products of random matrices (see [18] for an overview of rigorous results on this subject).

We will denote averages with respct to this invariant distribution by $\langle\langle \ldots \rangle\rangle$. In particular the free energy itself is given by such an average, namely

$$-\beta F = -\beta J + \langle\langle \ln(\rho_n + e^{2\beta J}) \rangle\rangle. \tag{2.5}$$

It will turn out to be convenient to rewrite the basic formulae (2.4) and (2.5) using different choices of variables. For the sake of consistency with [1–3], we introduce the parameters w, λ, μ and y_0. These dimensionless quantities are defined as follows

$$\begin{aligned} w &= [2\sinh(2\beta J)]^{1/2} \qquad & \lambda &= 2\beta H \\ e^{-2\mu} &= \tanh(\beta J) \qquad & y_0 &= -J/H. \end{aligned} \tag{2.6}$$

We perform first the change of variable

$$V_n = e^{-\beta J} \left(e^{2\beta J} - \frac{e^{2\beta h_n}}{\rho_n} \right). \tag{2.7}$$

In terms of the V_n, the recursion (2.4) takes the form

$$V_n = \frac{w^2 e^{2\beta h_{n-1}}}{e^{\beta J} \left(1 + e^{2\beta h_{n-1}}\right) - V_{n-1}} \tag{2.8}$$

and the free energy can be expressed as

$$-\beta F = -\beta J + \langle\langle \ln\left[e^{2\beta J} + e^{\beta J - 2\beta h_n} \left(e^{\beta J} - V_n\right)\right] \rangle\rangle \tag{2.9}$$

where $\langle\langle \ldots \rangle\rangle$ denotes an average with respect to the stationary distribution of the random variables V_n.

We will also use the following second change of variables

$$Z_n = \frac{1 - \rho_n}{1 + \rho_n}. \tag{2.10}$$

In terms of these new variables, the recursion formula (2.4) reads

$$Z_n = \frac{Z_{n-1} e^{-2\mu} - t_n}{1 - Z_{n-1} t_n e^{-2\mu}} \tag{2.11}$$

with the notation

$$t_n = \tanh(\beta h_n) \tag{2.12}$$

and the free energy is given by

$$-\beta F = \ln(2\cosh\beta J) + \left\langle\!\!\left\langle \ln\frac{1+\mathrm{e}^{-2\mu}Z_n}{1+Z_n} \right\rangle\!\!\right\rangle . \tag{2.13}$$

where $\langle\!\langle \ldots \rangle\!\rangle$ now denotes an average with respect to the stationary distribution of the random variables Z_n.

The general formalism recalled earlier will be used in the following, in the study of the random-field Ising chain with the 'exactly solvable' class of distributions (1.2) of the random fields in sections 3 to 5, and with the diluted binary distribution (1.3) in section 6.

3. Exact solution at finite temperature

This section is devoted to an exact evaluation of the quenched free energy F, at any finite inverse temperature β, when the probability density $R(h)$ of the random fields assumes the power-times-exponential form (1.2), with an arbitrary value of the integer parameter ν. As explained in the introduction, this class of 'exactly solvable' distributions generalizes those considered in our previous works [1–3], which correspond to $\nu = 1$ and $\nu = 2$.

Our analysis will follow the lines of [1–3], starting from equations (2.8) and (2.9). A convenient way of dealing with the invariant distribution of the random variables V_n is to consider its logarithmic transform

$$E(y) = \langle\!\langle \ln(V_n - y) \rangle\!\rangle \tag{3.1}$$

as an analytic function of the variable y, in the plane cut along the positive real axis, since the support of the invariant distribution can be shown to be the interval $I = [0, w^2\mathrm{e}^{-\beta J}]$.

It follows from the recursion formula (2.8) that $E(y)$ obeys the following functional equation

$$E(y) = \ln y + \beta F + \left\langle E\left[\mathrm{e}^{\beta J}\left(1+\mathrm{e}^{2\beta h_n}\right) - \frac{w^2}{y}\mathrm{e}^{2\beta h_n}\right]\right\rangle \tag{3.2}$$

where $\langle \ldots \rangle$ denotes an averaging with respect to the distribution $R(h_n)$ of the random field at site n. Notice that the free energy, which shows up as a constant term in the functional equation (3.2), can be expressed in terms of $E(y)$ itself, using equation (2.9).

With the probability distribution (1.2) of the random magnetic fields, equation (3.2) can be rewritten as

$$E(y) = \ln y + \beta F + pE_\nu(y) + rE(y_1) \tag{3.3}$$

where E_ν is one of the following functions, defined for any integer $m \geqslant 1$ by

$$E_m(y) = \tfrac{1}{2}\left[E_m^{(+)}(y) + E_m^{(-)}(y)\right]$$

$$E_m^{(+)}(y) = \int_0^{+\infty} \frac{x^{m-1}\mathrm{e}^{-x}}{(m-1)!} E\left[\phi(x,y)\right] \mathrm{d}x$$

$$E_m^{(-)}(y) = \int_{-\infty}^{0} \frac{(-x)^{m-1}\mathrm{e}^{x}}{(m-1)!} E\left[\phi(x,y)\right] \mathrm{d}x \tag{3.4}$$

and where

$$y_1 = 2\mathrm{e}^{\beta J} - \frac{w^2}{y} \qquad \phi(x,y) = \mathrm{e}^{\beta J}\left(1+\mathrm{e}^{\lambda x}\right) - \frac{w^2}{y}\mathrm{e}^{\lambda x} \tag{3.5}$$

with the notation of equation (2.6). The function $\phi(x,y)$ obeys the identity

$$\partial_x \phi = \Lambda(y)\partial_y \phi \tag{3.6}$$

where

$$\Lambda(y) = \lambda\left(\mathrm{e}^{\beta J}\frac{y^2}{w^2} - y\right). \tag{3.7}$$

Equation (3.6) enables one to show, by means of one integration by parts, that the functions $E_m^{(\pm)}(y)$ obey the following first-order differential equations

$$\pm L\, E_m^{(\pm)}(y) = E_m^{(\pm)}(y) - E_{m-1}^{(\pm)}(y) \tag{3.8}$$

where L denotes the differential operator

$$L = \Lambda(y)\partial_y \tag{3.9}$$

and with the convention that $E_m^{(\pm)}(y) = E(y_1)$ if $m = 0$, where y_1 has been defined in equation (3.5).

Using equation (3.8) repeatedly, we can prove the following relations

$$(1-L)^m E_m^{(+)}(y) = (1+L)^m E_m^{(-)}(y) = E(y_1)$$

$$2(1-L^2)^m E_m(y) = \left[(1-L)^m + (1+L)^m\right] E(y_1). \tag{3.10}$$

This last identity can be used to recast the central equation (3.3) in the form

$$(1-L^2)^\nu \left[E(y) - rE(y_1) - \ln y\right] = \beta F + \frac{p}{2}\left[(1+L)^\nu + (1-L)^\nu\right] E(y_1). \tag{3.11}$$

We now perform the following change of functions and of variable

$$E(y) = G(z) - \ln\left(1 - z\mathrm{e}^{2\mu}\right) \qquad E_m(y) = g_m(z) - \ln(1-z)$$

$$z = \frac{w - y\mathrm{e}^{-\mu}}{w - y\mathrm{e}^{\mu}}. \tag{3.12}$$

This manipulation turns out to be quite advantageous. Notice for instance that y_1 is mapped onto $z_1 = z\mathrm{e}^{-2\mu}$. Equation (3.11) becomes then

$$(1 - L^2)^\nu \left[G(z) - rG(z\mathrm{e}^{-2\mu}) - p\ln(1-z)\right]$$
$$= \beta F_{\mathrm{R}} + \frac{p}{2}\left[(1+L)^\nu + (1-L)^\nu\right]\left[G(z\mathrm{e}^{-2\mu}) - \ln(1-z)\right] \qquad (3.13)$$

where F_{R} denotes the 'random part' of the free energy, i.e. the difference between the free energy F and its value F_0 for the pure Ising chain, in the absence of the random fields

$$F_{\mathrm{R}} = F - F_0 \qquad \text{with} \qquad \beta F_0 = -\ln(2\cosh\beta J) \qquad (3.14)$$

and where the operator L now reads

$$L = \frac{\lambda}{2}(1 - z^2)\partial_z. \qquad (3.15)$$

It can be checked, using the changes of variables introduced in section 2, that the function $G(z)$ is, apart from an irrelevant additive constant, nothing else than the logarithmic transform of the stationary distribution of the variables Z_n, introduced in equation (2.10), namely

$$G(z) - G(0) = \langle\!\langle \ln(1 - zZ_n)\rangle\!\rangle. \qquad (3.16)$$

It is evident from the recursion (2.11) that the Z_n have absolute values bounded by unity. Moreover, since Z_n is an odd function of all the random magnetic fields $h_1, \ldots, h_n$, the invariant distribution of the Z_n shares the evenness of the distribution of the random fields. We can therefore complete equation (3.13), in analogy with [1–3], by the requirement that the function $G(z)$ be even in z, and analytic in the z-plane cut along the real axis from $-\infty$ to -1, and from $+1$ to $+\infty$.

If we expand the function $G(z)$ as a power series in z, equation (3.13) yields a $(2\nu + 1)$-term recursion relation for the coefficients of this Taylor expansion, which determines them in a unique way, once boundary conditions have been taken care of in an appropriate way.

Owing to the complexity of the recursion relation just mentioned, we prefer to split it into several coupled simpler equations. To do so, we also introduce the Taylor expansions of the functions $g_m(z)$ defined in equation (3.12), for $1 \leqslant m \leqslant \nu$. More precisely, for the sake of further convenience, we set

$$G(z) = G(0) - \sum_{k(\text{even})\geqslant 2} C_k \frac{z^k}{k} \qquad g_m(z) = g_m(0) - \sum_{k(\text{even})\geqslant 2} d_{m,k}\frac{z^k}{k} \qquad (3.17)$$

where the sums only involve even values of the index k, because of the evenness of the functions considered. As a consequence of the property (3.16), we have

$$C_k = \langle\!\langle Z_n^k \rangle\!\rangle. \qquad (3.18)$$

In terms of the functions $G(z)$ and $g_m(z)$, equation (3.3) assumes the form

$$G(z) = \beta F_{\mathrm{R}} + p g_\nu(z) + rG(z\mathrm{e}^{-2\mu}). \qquad (3.19)$$

On the other hand, by iterating equation (3.8), we obtain the following system of differential equations

$$L^2 g_m(z) = -\frac{\lambda^2}{4}(1-z^2) + \left| \begin{array}{ll} g_m(z) - 2g_{m-1}(z) + g_{m-2}(z) & (3 \leqslant m \leqslant \nu) \\ g_2(z) - 2g_1(z) + G(z\mathrm{e}^{-2\mu}) & (m=2) \\ g_1(z) - G(z\mathrm{e}^{-2\mu}) & (m=1) \end{array} \right. . \tag{3.20}$$

By inserting the expansions (3.17) into equation (3.20), we are left after some algebraic manipulations with ν coupled three-term recursion relations of the form

$$\frac{\lambda^2}{4} k\left[(k+1)d_{m,k+2} + (k-1)d_{m,k-2} - 2kd_{m,k}\right] = \Delta_{m,k} \qquad (k \geqslant 2) \tag{3.21}$$

where the right-hand sides read

$$\Delta_{m,k} = \left| \begin{array}{ll} d_{m,k} - 2d_{m-1,k} + d_{m-2,k} & (3 \leqslant m \leqslant \nu) \\ d_{2,k} - 2d_{1,k} + C_k\mathrm{e}^{-2k\mu} & (m=2) \\ d_{1,k} - C_k\mathrm{e}^{-2k\mu} & (m=1) \end{array} \right. \tag{3.22}$$

together with the boundary conditions

$$d_{m,0} = 1 \qquad \lim_{k\to\infty} d_{m,k} = 0 \qquad (1 \leqslant m \leqslant \nu). \tag{3.23}$$

Equation (3.19) is equivalent to the constraint

$$C_k = \frac{p}{1 - r\mathrm{e}^{-2k\mu}} d_{\nu,k} \tag{3.24}$$

whereas the quenched free energy of the model can be simply expressed as

$$\beta F_{\mathrm{R}} = -p\frac{\lambda^2}{4}\sum_{m=1}^{\nu}(\nu+1-m)(1-d_{m,2}). \tag{3.25}$$

The formulae (3.21)–(3.25) represent an exact solution of the problem, in the usual sense that the crucial step of integrating over the random fields has been performed exactly in an analytical way. These results generalize our previous works [1–3] to the class of distributions (1.2) of the random magnetic fields, where the power which multiplies the exponential is an arbitrary integer ν. For $\nu = 1$ and $\nu = 2$, the expressions derived in [1–3] are recovered.

Equations (3.21)–(3.24) can be solved numerically, and thus yield essentially exact values of the free energy, and thermodynamic functions, as long as the temperature is moderate. The numerical treatment of this exact solution consists in the following two steps.

(i) Determine an arbitrary basis of ν independent solutions $d^{(a)}_{m,k}$ $(1 \leqslant a \leqslant \nu)$ of the linear recursion equations (3.21)–(3.24) which go to zero for large k, according to the second of the boundary conditions (3.23). It can indeed be checked that this is always possible, and that these solutions fall off proportionally to $k^{-1/\lambda}$. From a practical viewpoint, one imposes that the solutions vanish identically for $k > k_{\max}$, where $k_{\max}$ is some large cutoff.

(ii) Determine the coefficients $N_{(a)}$ of the linear combination

$$d_{m,k} = \sum_{a=1}^{\nu} N_{(a)} d_{m,k}^{(a)} \tag{3.26}$$

in such a way that $d_{m,0} = 1$, for $1 \leqslant m \leqslant \nu$, according to the first of the boundary conditions (3.23). These are ν linear equations for ν unknown quantities, so that the problem has a unique solution, provided the linear system under consideration is regular: this is always the case for real positive temperatures.

At low temperatures, the number of terms in expansions (3.17), which are necessary to get a reasonably accurate result, diverges as $k_{\max} \sim \exp(2\beta J)$, so that a qualitatively different analysis is needed. This is the main subject of sections 4 and 5.

4. Low-temperature behaviour: technicalities

In this section, we analyse in detail the low-temperature behaviour of the free energy of the random-field Ising chain, with the distribution (1.2) of the random fields, starting from the finite-temperature exact solution derived in section 3. This analysis follows the lines of our previous works [1–3].

4.1. General approach

At low temperature, the sequences C_k and $d_{m,k}$ introduced in section 3 have their relevant variations for typical values of the index k of the order of $\exp(2\beta J)$. We are thus led to introduce the continuous scaled variable

$$y = \frac{1}{\lambda} \ln(2k\mu) = y_0 + \frac{1}{\lambda} \ln(2k) \tag{4.1}$$

where y_0 and λ have been introduced in equation (2.6). The last equality of equation (4.1) is valid up to exponentially small terms in temperature, of order $\exp(-4\beta J)$, which will be neglected throughout the following analysis.

The key idea of the approach is that the difference equations (3.21) and (3.22) can be replaced by coupled differential equations for unknown functions $C(y)$ and $d_m(y)$, defined as being the scaled low-temperature limits of the sequences C_k and $d_{m,k}$ under the change of variable (4.1). By expanding equation (3.21) for large k, one realizes that the left-hand side of that equation is asymptotically equal to d''_m, where the accents denote differentiations with respect to the variable y. This estimate holds up to terms of relative order $1/k$, indicating thus that special care will have to be taken in doing the matching at $y = y_0$ with the solution of the difference equations (3.21) for large but finite values of the index k.

This procedure leads to ν coupled second-order differential equations, namely

$$\left| \begin{array}{ll} d''_m(y) = d_m(y) - 2d_{m-1}(y) + d_{m-2}(y) & \qquad (3 \leqslant m \leqslant \nu) \\ d''_2(y) = d_2(y) - 2d_1(y) + [1 - u(y)]\, d_\nu(y) & \\ d''_1(y) = d_1(y) - [1 - u(y)]\, d_\nu(y) & \end{array} \right. \tag{4.2}$$

where the function $u(y)$ is defined as

$$u(y) = \frac{1 - v(y)}{1 - rv(y)} \qquad \text{with} \qquad v(y) = \exp(-e^{\lambda y}). \tag{4.3}$$

In the low-temperature limit, we have $v(y) \approx \theta(-y)$ and $u(y) \approx \theta(y)$, where $\theta(y)$ denotes Heaviside's step function, equal to 1 (respectively 0) when y is positive (respectively negative). This holds true except in a small region of size $1/\lambda$ around the origin $y = 0$, which will play an essential role in the analysis.

The technical development which follows has been split into four points, for the sake of clarity.

4.1.1. Solution for $y > 0$. In this case, we can replace the function $u(y)$ by unity in the right-hand side of the differential equations (4.2). These equations are then very easily solved by recursion over increasing values of m, starting with the equation for d_1. We thus obtain

$$d_m(y) = e^{-y} \sum_{n=0}^{m-1} P_{m-n} \frac{y^n}{n!} \tag{4.4}$$

where the P_n $(1 \leqslant n \leqslant \nu)$ are ν integration constants, which depend *a priori* on temperature, and which so far remain unknown.

4.1.2. Solution for $y_0 < y < 0$. In this second domain, the function $u(y)$ is to be replaced by zero in the right-hand side of equation (4.2). The system of differential equations thus obtained is less easy to solve than in the previous case. If we forget for a while about the boundary conditions, the set of solutions to that system is clearly a linear space with dimension 2ν.

Let us look for elementary solutions of the system (4.2) such that

$$d''_m(y) = \omega^2 d_m(y). \tag{4.5}$$

A basis of such functions assumes the form

$$d_m(y) = b_m e^{\omega y} \tag{4.6}$$

where ω is determined up to a sign. We obtain after some algebra

$$b_m = (1+\omega)^{-m} + (1-\omega)^{-m} \tag{4.7}$$

and we find that ω has to obey the consistency equation

$$(1+\omega)^{-\nu} + (1-\omega)^{-\nu} = 2. \tag{4.8}$$

Equation (4.8), which can be recast as

$$(1-\omega^2)^\nu - \tfrac{1}{2}\left[(1+\omega)^\nu + (1-\omega)^\nu\right] = -\omega^2 \Phi_\nu(\omega^2) = 0 \tag{4.9}$$

is therefore the characteristic (or secular) equation of the problem. Φ_ν is a polynomial of degree $\nu - 1$ in its argument ω^2. Hence it has exactly $2(\nu - 1)$ complex zeros, denoted by $\pm\omega_a$ $(1 \leqslant a \leqslant \nu - 1)$ in the following. For the sake of definiteness, we denote by ω_a the $(\nu - 1)$ roots of equation (4.9) *with positive real parts.* We will comment on the occurrence of these complex roots in the discussion, and put them in perspective with the results of [1, 8].

The first non-trivial case is $\nu = 2$, where we have $\Phi_2(\omega^2) = 3 - \omega^2$, so that the unique characteristic value reads $\omega_1 = \sqrt{3}$: this number plays an important part in the solution presented in [3]. The next few polynomials read $\Phi_3(\omega^2) = 6 - 3\omega^2 + \omega^4$, $\Phi_4(\omega^2) = 10 - 5\omega^2 + 4\omega^4 - \omega^6$, and so on. The behaviour of the roots ω_a for large ν will be discussed in full detail in the beginning of section 6.2.

We have thus found $2(\nu - 1)$ linearly independent solutions to the system (4.2), obtained by setting either $\omega = \omega_a$ or $\omega = -\omega_a$ in equations (4.6) and (4.7). The last two solutions, which read $d_m(y) = 1$ and $d_m(y) = y$, independently of m, correspond formally to the double root $\omega = 0$ of equation (4.9).

As a consequence of this discussion, the solution of the system (4.2) in the interval $y_0 < y < 0$ reads

$$d_m(y) = Ay + B + \sum_{a=1}^{\nu-1} b_{a,m}\left(E_a \mathrm{e}^{\omega_a y} + F_a \mathrm{e}^{-\omega_a y}\right) \tag{4.10}$$

with

$$b_{a,m} = (1 + \omega_a)^{-m} + (1 - \omega_a)^{-m} \tag{4.11}$$

we have thus in particular $b_{a,\nu} = 2$. A, B and the E_a, F_a $(1 \leqslant a \leqslant \nu - 1)$ are 2ν (temperature-dependent) constants yet to be determined.

4.1.3. Matching at $y = y_0$. The aim of this third point is to study in more detail how the solution (4.10), obtained by going from the integer index k to the continuous scaled variable y, matches for $y \to y_0$ with the solution of the full difference equations (3.21) and (3.22), for values of k in the range $1 \leqslant k \leqslant 1/\mu$.

One powerful way of dealing with this question consists in solving directly equation (3.13) for $\mu = 0$. Under this condition, equation (3.13) looses its non-local functional character, and becomes an ordinary differential equation of the form

$$\mathcal{L}\left[G(z) - \ln(1 - z)\right] = \beta F_{\mathrm{R}}/p \tag{4.12}$$

where the differential operator $\mathcal{L}$ reads

$$\mathcal{L} = (1 - L^2)^\nu - \tfrac{1}{2}\left[(1 + L)^\nu + (1 - L)^\nu\right] = -L^2\Phi_\nu(L^2). \tag{4.13}$$

In order to solve equation (4.12), it is advantageous to introduce the linearizing variable

$$t = \frac{1}{\lambda}\ln\frac{1 + z}{1 - z} \tag{4.14}$$

in terms of which we have simply

$$L = \partial_t. \tag{4.15}$$

Equation (4.12) is therefore very easily solved. The general solution of the associated homogeneous equation is an arbitrary linear combination of the functions 1, t and

$\exp(\pm\omega_a t)$ $(1 \leqslant a \leqslant \nu - 1)$. Moreover, we notice that a particular solution of the inhomogeneous equation (4.12) reads $G_0(z) = -\beta F_{\rm R} t^2/[p\nu(\nu+1)]$, since the constant coefficient of the polynomial Φ_ν is equal to $\nu(\nu+1)/2$.

Going back to the z variable, and imposing the requirement, discussed below equation (3.16), that $G(z)$ be an even function, we are left with the following expression for $G(z)$ in the $\mu = 0$ limit

$$G(z) = G(0) + \frac{1}{2}\ln(1-z^2) - \frac{\beta F_{\rm R}}{p\nu(\nu+1)\lambda^2}\left(\ln\frac{1+z}{1-z}\right)^2 + \sum_{a=1}^{\nu-1}\delta_a\left[\left(\frac{1+z}{1-z}\right)^{\omega_a/\lambda} + \left(\frac{1-z}{1+z}\right)^{\omega_a/\lambda}\right] \tag{4.16}$$

where the δ_a are arbitrary integration constants. The final step consists in expanding equation (4.16) in a Taylor series in z. The outcome reads, for large k and with the notation of equation (3.17)

$$C_k \approx 1 + \frac{4\beta F_{\rm R}}{p\nu(\nu+1)\lambda^2}(\ln 2k + \gamma_{\rm E}) - 2\sum_{a=1}^{\nu-1}\delta_a\left[\frac{(2k)^{\omega_a/\lambda}}{\Gamma(\omega_a/\lambda)} + \frac{(2k)^{-\omega_a/\lambda}}{\Gamma(-\omega_a/\lambda)}\right]. \tag{4.17}$$

Here and throughout the following, $\gamma_{\rm E} \approx 0.57721$ denotes Euler's constant. This large-k behaviour must match the expression (4.10) for $m = \nu$, since C_k and $d_{\nu,k}$ coincide, up to exponentially small terms, by virtue of equation (3.24). By expressing this matching, we obtain the following conditions between the constants which enter equation (4.10)

$$B = 1 + (\gamma_{\rm E}/\lambda - y_0)A \tag{4.18}$$

$$\Gamma(1-\omega_a/\lambda)F_a{\rm e}^{-\omega_a y_0} = -\Gamma(1+\omega_a/\lambda)E_a{\rm e}^{\omega_a y_0} = \omega_a\delta_a/\lambda \tag{4.19}$$

and we can rewrite the free energy of the model as

$$F_{\rm R} = \frac{p}{2}\nu(\nu+1)H\ A. \tag{4.20}$$

4.1.4. Matching at $y = 0$. A systematic and convenient way of dealing with the solution of the system (4.2) for y close to the origin is to make use of Laplace transforms, in analogy with our previous works. We define the Laplace transforms $D_m(z)$ of the functions $d_m(y)$ by

$$D_m(z) = \int_{y_0}^{+\infty} {\rm e}^{zy} d_m(y)\ {\rm d}y \qquad (\mathrm{Re}\ z < 1) \tag{4.21}$$

and the Laplace transform of the function $u(y)$ defined in equation (4.3) in a slightly different way

$$\mathcal{U}(z) = \int_{-\infty}^{+\infty} {\rm e}^{-zy} u(y)\ {\rm d}y \qquad (0 < \mathrm{Re}\ z < \lambda). \tag{4.22}$$

An elementary calculation yields

$$\mathcal{U}(z) = \frac{1}{z}\Gamma\left(1-\frac{z}{\lambda}\right) f\left(\frac{z}{\lambda}\right) \qquad \text{with} \qquad f(s) = p\sum_{n\geqslant 1} r^{n-1} n^{s}. \quad (4.23)$$

Notice that the function $f(s)$ is analytic in the whole complex s-plane. We also need the Laplace transform of the second derivative of the $d_m(y)$, for which a direct evaluation yields

$$\int_{y_0}^{+\infty} \mathrm{e}^{zy} d''_m(y)\ \mathrm{d}y = z^2 D_m(z) + S_m(z) \quad (4.24)$$

where $S_m(z)$ represents the boundary terms, which can be evaluated by means of equation (4.10), namely

$$\begin{aligned} S_m(z) &= \mathrm{e}^{zy_0}\left[z d_m(y_0) - d'_m(y_0)\right] \\ &= \mathrm{e}^{zy_0}\left[-A - \sum_{a=1}^{\nu-1} b_{a,m}\omega_a \left(E_a \mathrm{e}^{\omega_a y_0} - F_a \mathrm{e}^{-\omega_a y_0}\right)\right. \\ &\quad \left. + z\left\{A y_0 + B + \sum_{a=1}^{\nu-1} b_{a,m}\left(E_a \mathrm{e}^{\omega_a y_0} + F_a \mathrm{e}^{-\omega_a y_0}\right)\right\}\right]. \end{aligned} \quad (4.25)$$

The system (4.2) is then equivalent to

$$\begin{aligned} S_m(z) &= (1-z^2)D_m(z) - 2D_{m-1}(z) + D_{m-2}(z) \qquad (3 \leqslant m \leqslant \nu) \\ S_2(z) + \Delta(z) &= (1-z^2)D_2(z) - 2D_1(z) + D_\nu(z) \\ S_1(z) - \Delta(z) &= (1-z^2)D_1(z) - D_\nu(z) \end{aligned} \quad (4.26)$$

where

$$\begin{aligned} \Delta(z) &= \int \frac{\mathrm{d}s}{2\pi\mathrm{i}} \mathcal{U}(s) D_\nu(z+s) \\ &= \int_{y_0}^{+\infty} \mathrm{e}^{zy} u(y) d_\nu(y)\ \mathrm{d}y \qquad (0 < \mathrm{Re}\ s < \lambda;\ \mathrm{Re}\,(z+s) < 1). \end{aligned} \quad (4.27)$$

Our next purpose is to extract from the system (4.26) expressions for the constants A, B, E_a and F_a, in terms of the function $\Delta(z)$ only, since this last quantity will turn out to have a simpler low-temperature expansion.

In order to evaluate A and B, we multiply the equation of (4.26) which involves S_m by $(\nu+1-m)$, we sum up the results, and we rearrange terms by means of the identity $\sum_{m=1}^{\nu}(\nu+1-m)b_{a,m} = 0$, valid for any value of the root label a. We are left with the estimate

$$\Delta(z) = \tfrac{1}{2}\nu(\nu+1)\left[(Ay_0+B)z - A\right]\mathrm{e}^{zy_0} + \mathrm{O}(z^2) \qquad (z \to 0) \quad (4.28)$$

from which we can extract easily

$$\Delta(0) = -\tfrac{1}{2}\nu(\nu+1)A \qquad \Delta'(0) = \tfrac{1}{2}\nu(\nu+1)B. \quad (4.29)$$

In order to evaluate the constants E_a and F_a, we perform a similar manipulation on the system (4.26), namely we multiply the equation involving S_m by $\left[(1+z)^{m-\nu-1}-(1-z)^{m-\nu-1}\right]$, and we sum these equations successively for $z=\omega_a$ and for $z=-\omega_a$. The condition $\Phi_\nu(\omega_a^2)=0$ allows the calculation to be simplified drastically, so that we are only left with

$$E_a = \frac{\Delta(-\omega_a)}{\nu\rho_a} \qquad F_a = -\frac{\Delta(\omega_a)}{\nu\rho_a} \tag{4.30}$$

with the notation $\rho_a=(1+\omega_a)^{-\nu-1}-(1-\omega_a)^{-\nu-1}$.

The expressions (4.29) and (4.30) allow us to rewrite the conditions (4.18) and (4.19) as

$$\Gamma(1+\omega_a/\lambda)\Delta(-\omega_a)\mathrm{e}^{\omega_a y_0} = \Gamma(1-\omega_a/\lambda)\Delta(\omega_a)\mathrm{e}^{-\omega_a y_0} \tag{4.31}$$

$$\Delta'(0)+(\gamma_{\mathrm{E}}/\lambda-y_0)\Delta(0)=\tfrac{1}{2}\nu(\nu+1) \tag{4.32}$$

and to express the free energy as

$$F_{\mathrm{R}} = -pH\,\Delta(0). \tag{4.33}$$

4.2. Low-temperature expansion

The purpose of this section is to show that the free energy $F=F_0+F_{\mathrm{R}}$ of the model admits an expansion in powers of temperature, of the form

$$F = E_0 - S_0\,T - \Gamma_0\,T^2/2 + \cdots. \tag{4.34}$$

The physical meaning of these coefficients is the following. E_0 and S_0 are the zero-temperature (ground-state) energy and entropy, respectively, whereas Γ_0 is the amplitude of the linear low-temperature behaviour $C(T)\approx\Gamma_0 T$ of the specific heat. In the following, one of our main goals will consist in evaluating the values of E_0, S_0 and Γ_0.

The expression (4.33) shows that it will be sufficient to know the low-temperature expansion of the function $\Delta(z)$. This expansion can be evaluated as follows.

We start by shifting the integration contour of equation (4.27) to the right of the pole at $z+s=1$, thus obtaining

$$\Delta(z) = \mathcal{R} + \int \frac{\mathrm{d}s}{2\pi\mathrm{i}}\mathcal{U}(s)D_\nu(z+s) \qquad (0<\mathrm{Re}\ s<\lambda;\ \mathrm{Re}\ (z+s)>1). \tag{4.35}$$

In order to determine the residue $\mathcal{R}$ of the integrand at $z+s=1$, we notice that, as a consequence of equation (4.4), the Laplace transform $D_\nu(z)$ has a polar part at $z=1$, of the form

$$D_\nu(z) \sim \sum_{m=0}^{\nu-1}\frac{P_{\nu-m}}{(1-z)^{m+1}} \qquad (z\to 1). \tag{4.36}$$

This expression easily yields the value of $\mathcal{R}$, namely

$$\mathcal{R} = \sum_{m=0}^{\nu-1}\frac{(-1)^m}{m!}P_{\nu-m}\mathcal{U}^{(m)}(1-z) \tag{4.37}$$

where the superscript (m) denotes an order of differentiation.

It can be argued, along the lines of our previous studies (see [1–3]), that the contour integral in the right-hand side of equation (4.35) is smaller than the residue $\mathcal{R}$ by at least a factor of T^3 at low temperature. The key observation is that $D_\nu(z)$ decreases at least as $1/z^2$ for Re z large positive, i.e. comparable with λ. As a consequence, the low-temperature expansion of $\Delta(z)$ coincides, including terms up to T^2 (i.e. in $1/\lambda^2$), with the expansion of the residue $\mathcal{R}$ itself.

The following expansions of the derivatives of the function $\mathcal{U}(z)$, up to and including terms proportional to $1/\lambda^2$, will be useful in the following

$$\begin{aligned}\mathcal{U}(z) &= \frac{1}{z} + \frac{1}{\lambda}(\gamma_{\mathrm{E}} + s_1) + \frac{z}{2\lambda^2}(\gamma_{\mathrm{E}}^2 + \pi^2/6 + 2\gamma_{\mathrm{E}}s_1 + s_2) + \ldots \\ \mathcal{U}'(z) &= -\frac{1}{z^2} + \frac{1}{2\lambda^2}(\gamma_{\mathrm{E}}^2 + \pi^2/6 + 2\gamma_{\mathrm{E}}s_1 + s_2) + \ldots \\ \mathcal{U}^{(m)}(z) &= \frac{(-1)^m m!}{z^{m+1}} + \ldots \qquad (m \geqslant 2).\end{aligned} \tag{4.38}$$

The parameters s_1 and s_2 depend only on the impurity concentration p, according to

$$s_k = f^{(k)}(0) = p \sum_{n \geqslant 1} r^{n-1} (\ln n)^k. \tag{4.39}$$

These quantities vanish linearly in $r = 1 - p$ for p close to unity. Their small-p behaviour can be obtained by replacing the sum in equation (4.39) by an integral. This leads us to the following logarithmic behaviour

$$s_k \approx |\ln p|^{\,k} - k\gamma_{\mathrm{E}} |\ln p|^{k-1} + \ldots. \tag{4.40}$$

We have thus derived the low-temperature expansions, including terms up to order T^2, of all the relevant quantities, except for the ν constants P_m $(1 \leqslant m \leqslant \nu)$. The derivation of the low-temperature expansion of these quantities is indeed a more difficult task, which will now be presented in several steps.

4.2.1. Zeroth order in T: the ground-state energy E_0. We start the detailed presentation of the low-temperature analysis with the zeroth order in temperature, which yields the ground-state energy E_0. We assume therefore that all the quantities under consideration have finite zero-temperature limits.

To this leading order of approximation, equations (4.35), (4.37) and (4.38) lead to

$$\Delta(z) = \sum_{m=0}^{\nu-1} \frac{P_{\nu-m}}{(1-z)^{m+1}} \tag{4.41}$$

where it is understood that the P_m stand for their zero-temperature values.

We choose to set, for the sake of further convenience,

$$\Delta(z) = \frac{\mathcal{N}_0 Q(z)}{(1-z)^\nu} \tag{4.42}$$

where Q is a normalized (monic) polynomial of degree $\nu - 1$, of the form

$$Q(z) = \sum_{m=0}^{\nu-1} Q_m z^m \qquad \text{with} \qquad Q_{\nu-1} = 1 \tag{4.43}$$

and where $\mathcal{N}_0 = (-1)^{\nu-1} P_{\nu-1}$. This normalization constant will play no essential part in the following.

The conditions (4.31) assume the form

$$(1+\omega_a)^\nu \mathrm{e}^{-\omega_a y_0} Q(\omega_a) = (1-\omega_a)^\nu \mathrm{e}^{\omega_a y_0} Q(-\omega_a) \qquad (1 \leqslant a \leqslant \nu-1) \tag{4.44}$$

whereas equations (4.32) and (4.33) allow us to express the ground-state energy as

$$E_0 = -J - \frac{p}{2}\nu(\nu+1)\frac{H^2}{J+\alpha_0 H} \qquad \text{with} \qquad \alpha_0 = \nu + \frac{Q_1}{Q_0}. \tag{4.45}$$

The only non-trivial ingredient in this exact result for the ground-state energy is the dimensionless quantity α_0. In order to evaluate it, one has to find a polynomial Q of degree $(\nu-1)$, which obeys the conditions (4.44) and the normalization (4.43). We recall that the ω_a are the complex roots of the secular equation (4.9), with strictly positive real parts.

Before going into more detail, we first derive analogous results concerning the zero-temperature entropy, and the specific heat amplitude.

4.2.2. First order in T: the zero-temperature entropy S_0. To this next order of approximation, equations (4.35), (4.37) and (4.38) lead to

$$\Delta(z) = \sum_{m=0}^{\nu-1} \frac{P_{\nu-m}}{(1-z)^{m+1}} + \frac{\gamma_{\mathrm{E}} + s_1}{\lambda} P_\nu \tag{4.46}$$

where the coefficients P_m themselves depend *a priori* on temperature.

As a consequence, the product $(1-z)^\nu \Delta(z)$ is a polynomial of degree ν, and therefore contains $\nu+1$ coefficients. Equation (4.46) expresses these coefficients in terms of the ν quantities P_m $(1 \leqslant m \leqslant \nu)$. Hence we can conclude that equation (4.46) imposes *one condition* among those coefficients.

If we choose to set, for the sake of convenience,

$$\Delta(z) = \frac{\mathcal{N}(\lambda)}{(1-z)^\nu}\left[\left(1 - \frac{\gamma_{\mathrm{E}}}{\lambda} z\right) Q(z) - \frac{s_1}{\lambda} R(z)\right] \tag{4.47}$$

where R is a polynomial of degree ν, and $\mathcal{N}(\lambda)$ an inessential temperature-dependent normalization constant, then the condition expressed by equation (4.46) is fulfilled, up to and including terms of order $1/\lambda$, if we require that $R(z)$ be of the form

$$R(z) = \sum_{m=1}^{\nu} R_m z^m \qquad \text{with} \qquad R_\nu = 1. \tag{4.48}$$

Notice that the definition (4.48) implies in particular that $R(0) = 0$.

On the other hand, the conditions (4.31), expanded in an appropriate way, yield homogeneous equations for $R(z)$ which are fully analogous to equation (4.44), namely

$$(1+\omega_a)^\nu \mathrm{e}^{-\omega_a y_0} R(\omega_a) = (1-\omega_a)^\nu \mathrm{e}^{\omega_a y_0} R(-\omega_a) \qquad (1 \leqslant a \leqslant \nu-1). \tag{4.49}$$

Finally, equations (4.32) and (4.33) allow us to express the zero-temperature entropy as

$$S_0 = \frac{p\, s_1}{4}\nu(\nu+1)\left(\frac{H}{J+\alpha_0 H}\right)^2 \alpha_1 \qquad \text{with} \qquad \alpha_1 = \frac{R_1}{Q_0}. \tag{4.50}$$

Besides α_0, which showed up already in the expression (4.45) of the ground-state energy, the evaluation of the zero-temperature entropy involves thus one new dimensionless quantity α_1, which is related to both polynomials Q and R.

4.2.3. Second order in T: the specific heat amplitude Γ_0. To this order of approximation, equations (4.35, 37, 38) lead to

$$\Delta(z) = \sum_{m=0}^{\nu-1} \frac{P_{\nu-m}}{(1-z)^{m+1}} + \frac{\gamma_{\rm E}+s_1}{\lambda} P_\nu + \frac{1}{2\lambda^2}(\gamma_{\rm E}^2+\pi^2/6+2\gamma_{\rm E}s_1+s_2)\left[(1-z)P_\nu - P_{\nu-1}\right]. \tag{4.51}$$

In analogy with the discussion below equation (4.46), we can conclude that equation (4.51) now imposes *two conditions* on the function $\Delta(z)$. Surprisingly enough, it turns out that these conditions can be met, up to and including the second order in $1/\lambda$, by the expression for $\Delta(z)$ given later, which only involves the polynomials $Q(z)$ and $R(z)$ introduced previously, without any need for defining any new unknown polynomial.

Indeed, if we choose to set

$$\Delta(z) = \frac{\mathcal{N}(\lambda)}{(1-z)^\nu}\left\{\left[1-\frac{\gamma_{\rm E}}{\lambda}z+(\gamma_{\rm E}^2-\pi^2/6)\frac{z^2}{2\lambda^2}\right]Q(z)-\frac{s_1}{\lambda}(1-\frac{\gamma_{\rm E}}{\lambda}z)R(z)+\frac{1}{\lambda^2}S(z)\right\} \tag{4.52}$$

then $S(z)$ has to be a polynomial of degree $(\nu+1)$. By expanding equation (4.31), it can be shown that $S(z)$ obeys the very same homogeneous conditions as $Q(z)$ and $R(z)$ [see equations (4.44) and (4.49)]. One can therefore look for a solution in the form $S(z) = a_1 z^2 Q(z) + a_2 R(z)$. The (temperature-independent) coefficients a_1 and a_2 can be determined by matching equations (4.51) and (4.52).

After some algebra, we are left with the following final expression for the amplitude Γ_0 of the linear low-temperature specific heat

$$\Gamma_0 = \frac{p}{4}\nu(\nu+1)\left\{\frac{H\alpha_1}{(J+\alpha_0 H)^2}\left[(s_2+\pi^2/6)\alpha_2 - s_1^2\alpha_3\right]+\frac{(H\alpha_1 s_1)^2}{(J+\alpha_0 H)^3}\right\}. \tag{4.53}$$

This expression involves two new quantities, α_2 and α_3, defined by

$$\alpha_2 = \nu + Q_{\nu-2} \qquad \alpha_3 = \nu + R_{\nu-1}. \tag{4.54}$$

To summarize this section, we have derived exact analytical expressions for the ground-state energy, the zero-temperature entropy, and the amplitude of the linear low-temperature specific heat in terms of four non-trivial dimensionless parameters α_0, α_1, α_2 and α_3 [see equations (4.45), (4.50), (4.53) and (4.54)]. The following sections will be devoted to more explicit expressions of the general results derived earlier, in several limiting cases, and other specific situations of interest.

5. Low-temperature behaviour: results

This section presents a more detailed discussion of the general exact results derived in section 4, concerning the low-temperature behaviour of the random-field Ising chain, where the random magnetic fields have the power-times-exponential distribution (1.2). We will consider first the generic situation, and study then some limiting and other specific cases of physical interest.

5.1. The generic situation

We recall first the general expressions, derived in section 4, for the ground-state energy E_0, the zero-temperature entropy S_0, and the amplitude Γ_0 of the linear low-temperature specific heat

$$E_0 = -J - \frac{p}{2}\nu(\nu+1)\frac{H^2}{J+\alpha_0 H} \tag{5.1a}$$

$$S_0 = \frac{p\, s_1}{4}\nu(\nu+1)\left(\frac{H}{J+\alpha_0 H}\right)^2 \alpha_1 \tag{5.1b}$$

$$\Gamma_0 = \frac{p}{4}\nu(\nu+1)\left\{\frac{H\alpha_1}{(J+\alpha_0 H)^2}\left[(s_2+\pi^2/6)\alpha_2 - s_1^2\alpha_3\right] + \frac{(H\alpha_1 s_1)^2}{(J+\alpha_0 H)^3}\right\}. \tag{5.1c}$$

These formulae involve four dimensionless quantities, denoted by α_0, α_1, α_2 and α_3, which contain the whole non-triviality of our exact solution. These numbers are defined through

$$\alpha_0 = \nu + \frac{Q_1}{Q_0} \qquad \alpha_1 = \frac{R_1}{Q_0} \qquad \alpha_2 = \nu + Q_{\nu-2} \qquad \alpha_3 = \nu + R_{\nu-1} \tag{5.2}$$

in terms of some of the coefficients of two polynomials $Q(z)$ and $R(z)$, of respective degrees $(\nu-1)$ and ν, defined in equations (4.43) and (4.48), and subjected to the conditions (4.44) and (4.49).

More precisely, equations (4.44) and (4.49) can be recast in the following form

$$\begin{gathered}\sum_{0\leqslant m(\mathrm{even})\leqslant \nu-1} Q_m\omega_a^m + \sum_{1\leqslant m(\mathrm{odd})\leqslant \nu-1} Q_m\omega_a^m x_a = 0\\ \sum_{2\leqslant m(\mathrm{even})\leqslant \nu} R_m\omega_a^m + \sum_{1\leqslant m(\mathrm{odd})\leqslant \nu} R_m\omega_a^m x_a = 0\end{gathered} \tag{5.3}$$

with the notation

$$x_a = \frac{1+t_a}{1-t_a} \qquad t_a = \left(\frac{1-\omega_a}{1+\omega_a}\right)^\nu \exp(-2\omega_a J/H). \tag{5.4}$$

Since the index a varies between 1 and $(\nu-1)$, the system (5.3) consists of $2(\nu-1)$ linear and homogeneous equations for the 2ν unknown coefficients Q_m and R_m. Since these quantities are also subjected to the conditions $R_\nu = Q_{\nu-1} = 1$, they are thus determined entirely. The Q_m and R_m can be expressed as ratios of determinants, which are generalizations of the famous van der Monde polynomials, for which we have unfortunately not found any appealing closed-form expression.

The general structure of the solution is, nevertheless, revealed in a clear fashion: the parameters α_0, α_1, α_2 and α_3 which enter the results (5.1) are complicated rational functions of all the x_a. Their dependence on the strength H of the random fields therefore involves, in a rational way, terms of the form $\exp(-2\omega_a J/H)$. The occurrence of these exponentials will be commented on in the discussion.

5.2. The first few values of ν

Let us now express our results in fully explicit form for the first few values of the integer ν.

5.2.1. $\nu = 1$. This case was solved for the first time in [1]. Within the present formalism, the problem looks simple, since the system (5.3) is just empty. The only non-vanishing coefficients are $Q_0 = R_1 = 1$, so that we are left with

$$\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 1. \tag{5.5}$$

By inserting these expressions into equation (5.1), with $\nu = 1$, we recover easily the results of [1] and [2].

5.2.2. $\nu = 2$. In this case, which was already solved in [3], there is only one characteristic value, $\omega_1 = \sqrt{3}$, as mentioned in section 4.1.2. The associated x-parameter reads

$$x_1 = \frac{1 + (2 - \sqrt{3})^2 \exp(-2\sqrt{3}\ J/H)}{1 - (2 - \sqrt{3})^2 \exp(-2\sqrt{3}\ J/H)} = \frac{2 + \sqrt{3}\tanh(\sqrt{3}\ J/H)}{\sqrt{3} + 2\tanh(\sqrt{3}\ J/H)}. \tag{5.6}$$

The solution of the system (5.3) is then

$$Q_0 = -x_1\sqrt{3} \qquad R_1 = -\sqrt{3}/x_1 \tag{5.7}$$

so that we are left with

$$\alpha_0 = 2 - \frac{1}{x_1\sqrt{3}} \qquad \alpha_1 = \frac{1}{x_1^2} \qquad \alpha_2 = 2 - x_1\sqrt{3} \qquad \alpha_3 = 2 - \frac{\sqrt{3}}{x_1}. \tag{5.8}$$

By inserting these expressions, together with $\nu = 2$, into equation (5.1), we reproduce the main results of [3].

5.2.3. $\nu = 3$. In this case, the characteristic values ω_a are two complex conjugate numbers, namely $\omega_{1,2} = 1.405\,256\,16 \pm 0.689\,017\,32i$. The solution of the system (5.3), which reads

$$\begin{aligned} Q_0 &= \frac{(x_2\omega_1 - x_1\omega_2)\omega_1\omega_2}{x_1\omega_1 - x_2\omega_2} \qquad & Q_1 &= \frac{\omega_2^2 - \omega_1^2}{x_1\omega_1 - x_2\omega_2} \\ R_1 &= \frac{(x_1\omega_1 - x_2\omega_2)\omega_1\omega_2}{x_2\omega_1 - x_1\omega_2} \qquad & R_2 &= \frac{(\omega_2^2 - \omega_1^2)x_1 x_2}{x_2\omega_1 - x_1\omega_2} \end{aligned} \tag{5.9}$$

yields the following expressions

$$\begin{aligned} \alpha_0 &= 3 + \frac{\omega_2^2 - \omega_1^2}{(x_2\omega_1 - x_1\omega_2)\omega_1\omega_2} \qquad & \alpha_1 &= \left(\frac{x_1\omega_1 - x_2\omega_2}{x_2\omega_1 - x_1\omega_2}\right)^2 \\ \alpha_2 &= 3 + \frac{\omega_2^2 - \omega_1^2}{x_1\omega_1 - x_2\omega_2} \qquad & \alpha_3 &= 3 + \frac{(\omega_2^2 - \omega_1^2)x_1 x_2}{x_2\omega_1 - x_1\omega_2}. \end{aligned} \tag{5.10}$$

These last equations illustrate the general structure of the outcome of our exact solution, for a generic value of the integer ν.

5.3. The small-H limit

The behaviour of E_0, S_0 and Γ_0 in the regime where $H \leqslant J$, for a fixed value of the integer ν, is simple to derive from the earlier formalism.

Indeed, the quantities $\mathrm{e}^{\omega_a y_0}$ which enter the conditions (4.44) and (4.49) are exponentially small in the ratio $H/J = -y_0$, since all the characteristic values ω_a have positive real parts.

In a first step of the analysis, forgetting about exponentially small corrections, we can rewrite the conditions (4.44) and (4.49) as $Q(\omega_a) = R(\omega_a) = 0$. The latter equations admit the simple factorized solution

$$Q(z) = \prod_{a=1}^{\nu-1}(z-\omega_a) \qquad R(z) = zQ(z) \tag{5.11}$$

from which we can derive the following expressions

$$\alpha_0 = \nu - \Omega_{(-1)} \qquad \alpha_1 = 1 \qquad \alpha_2 = \alpha_3 = \nu - \Omega_{(1)} \tag{5.12}$$

with the definition

$$\Omega_{(n)} = \sum_{a=1}^{\nu-1} \omega_a^n \tag{5.13}$$

where n is any (positive or negative) integer. The quantities $\Omega_{(n)}$ only depend on the integer ν. Their large-ν behaviour will be analysed in section 6.2.

It turns out that the leading $H \to 0$ results (5.12) can be improved by means of a systematic perturbative expansion in powers of the variables t_a, defined in equation (5.4). In terms of these quantities, equations (4.44) and (4.49) can be rewritten in the simpler form

$$Q(\omega_a) = t_a Q(-\omega_a) \qquad R(\omega_a) = t_a R(-\omega_a) \qquad (1 \leqslant a \leqslant \nu - 1). \tag{5.14}$$

It is advantageous to describe the polynomials $Q(z)$ and $R(z)$ in terms of their complex roots, namely

$$Q(z) = \prod_{a=1}^{\nu-1}(z-\omega_a-\epsilon_a) \qquad R(z) = z\prod_{a=1}^{\nu-1}(z-\omega_a-\eta_a) \tag{5.15}$$

where the shifts ϵ_a and η_a of the roots, with respect to the $H = 0$ limit (5.11), are assumed to go to zero with the field strength H. These shifts can be derived, to first order in the t_a, by inserting the form (5.15) into the conditions (5.14). We thus obtain

$$\epsilon_a = -\eta_a = 2\omega_a t_a \prod_{b\neq a} \frac{\omega_b+\omega_a}{\omega_b-\omega_a} + \mathcal{O}\left\{t_b^2\right\}. \tag{5.16}$$

These results allow us to obtain explicitly, again to first order in the parameters t_a, the corrections to the $H = 0$ expression (5.12), in the form

$$\begin{aligned} \alpha_0 &= \nu - \Omega_{(-1)} + \sum_{a=1}^{\nu-1}\frac{\epsilon_a}{\omega_a^2} \qquad & \alpha_1 &= 1 - 2\sum_{a=1}^{\nu-1}\epsilon_a \\ \alpha_2 &= \nu - \Omega_{(1)} - \sum_{a=1}^{\nu-1}\epsilon_a \qquad & \alpha_3 &= \nu - \Omega_{(1)} + \sum_{a=1}^{\nu-1}\epsilon_a. \end{aligned} \tag{5.17}$$

These leading correction terms are exponentially small in the field strength H, for any fixed value of the integer parameter ν. The large-ν regime, where the real parts of some of the ω_a go to zero, will be studied in section 6.2.

5.4. The large-H limit

The analysis of the asymptotic behaviour of our exact solution in the converse limiting situation $H \gg J$ is slightly more intricate.

If the strength H of the random fields is strictly infinite, we have $y_0 = -J/H = 0$, and the exponential factors in equations (4.44) and (4.49) can be forgotten. We start the analysis by noticing that, under these circumstances, the polynomials

$$\tilde{Q}_\infty(z) = \frac{(1-z)^\nu - 1}{z} \qquad \tilde{R}_\infty(z) = (1-z)^\nu \tag{5.18}$$

obey the conditions (4.44) and (4.49). The notations with a tilde are to remind us that these polynomials do not fulfil the various conditions required by the definitions (4.43) and (4.48). One has, for example, $\tilde{R}(0) \neq 0$. The main advantage of the polynomials defined in equation (5.18) consists in the following identities, which are simple consequences of the definition of the roots ω_a, and which will be used extensively hereafter

$$\begin{gathered}\frac{1}{\tilde{Q}_\infty(\omega_a)} - \frac{1}{\tilde{Q}_\infty(-\omega_a)} = -2\omega_a \qquad \frac{1}{\tilde{R}_\infty(\omega_a)} + \frac{1}{\tilde{R}_\infty(-\omega_a)} = 2 \\ \frac{\tilde{R}_\infty(\omega_a)}{\tilde{Q}_\infty(\omega_a)} = \frac{\tilde{R}_\infty(-\omega_a)}{\tilde{Q}_\infty(-\omega_a)} \qquad (1 \leqslant a \leqslant \nu - 1).\end{gathered} \tag{5.19}$$

The next step consists in considering equations (4.44) and (4.49) for a small but finite value of the ratio $y_0 = -J/H$, and in looking for solutions to these equations in the form of power series in y_0. It turns out that a formal solution to this problem can be easily derived to all orders of perturbation theory in y_0. Indeed, an explicit calculation of the first few terms of the y_0 expansion suggests that the generic terms sum up to the following scaling form

$$\tilde{Q}(z) = \tilde{Q}_\infty(z) - \frac{1}{z} Y(zy_0) \qquad \tilde{R}(z) = \tilde{R}_\infty(z) + Y(zy_0). \tag{5.20}$$

It is then easy to determine the function $Y(x)$, by inserting the ansatz (5.20) into equations (4.44) and (4.49). Using the identities (5.19), we are left with the following simple form for the scaling function $Y(x)$

$$Y(x) = \mathrm{e}^x \sinh x = \tfrac{1}{2}(\mathrm{e}^{2x} - 1) = \tfrac{1}{2} \sum_{n \geqslant 1} \frac{(2x)^n}{n!}. \tag{5.21}$$

The solutions (5.20) are only formal ones, since they involve arbitrarily large powers of the variable z, and are therefore not polynomials in z. It can nevertheless be argued that equations (5.20) and (5.21) do provide the correct y_0 expansion of the solution, up to all orders, as long as the powers of z involved do not exceed the degrees of $Q(z)$ and $R(z)$, i.e. $(\nu - 1)$ and ν, respectively. In other words, the naive form of perturbation theory just exposed does not predict the terms of order $y_0^{\nu+1}$ for the polynomials $\tilde{Q}(z)$ and $\tilde{R}(z)$, nor for the polynomials $Q(z)$ and $R(z)$, which will be constructed as linear combinations of $\tilde{Q}(z)$ and $\tilde{R}(z)$.

It turns out that the terms of order $y_0^{\nu+1}$ for the polynomial $\tilde{Q}(z)$ are needed, in order to derive the first non-trivial correction term in the large-H expansion of the ground-state energy E_0. In order to determine these leading *anomalous* terms, we set

$$\tilde{Q}(z) = \tilde{Q}_\infty(z) - \frac{1}{2z}\sum_{n=1}^{\nu}\frac{(2zy_0)^n}{n!} + y_0^{\nu+1}\eta(z) + \mathrm{O}(y_0^{\nu+2}) \tag{5.22}$$

where it is assumed that the terms of the y_0-expansion up to y_0^ν are correctly predicted by equations (5.20) and (5.21). The coefficient of order $y_0^{\nu+1}$, denoted by $\eta(z)$, is certainly different from the prediction of equations (5.20) and (5.21), which reads $\eta_0(z) = -(2z)^\nu/(\nu+1)!$, since the degree of $\tilde{Q}(z)$ is $(\nu-1)$. A careful investigation of the expansion of equation (4.44) as a power series in y_0, using the identities (5.19), shows that $\eta(z)$ has to obey the following condition

$$\frac{\eta(\omega_a) - \eta_0(\omega_a)}{\tilde{Q}_\infty(\omega_a)} = \frac{\eta(-\omega_a) - \eta_0(-\omega_a)}{\tilde{Q}_\infty(-\omega_a)} \qquad (1 \leqslant a \leqslant \nu - 1). \tag{5.23}$$

This equation can be solved by means of the last of the identities (5.19), which leads to the explicit solution

$$\eta(z) = \eta_0(z) + \frac{(-2)^\nu}{(\nu+1)!}\tilde{R}_\infty(z). \tag{5.24}$$

By inserting this expression into equation (5.22), we obtain a polynomial $\tilde{Q}(z)$, which is equal to $Q(z)$, up to a multiplicative constant. This polynomial yields, via equations (5.1a) and (5.2), the following expansion of the ground-state energy for $H \geqslant J$

$$E_0 = -p\nu H - rJ - \frac{pJ}{(\nu+1)!}\left(\frac{2J}{H}\right)^\nu + \dots . \tag{5.25}$$

The first two terms are in accord with the expected behaviour of the ground-state energy of the Ising chain in a very strong diluted random field. Indeed, each spin which feels a non-zero field is aligned with it, thus bringing the contribution $-\langle|h_n|\rangle = -p\nu H$ to E_0. The nature of the leading corrections present in equation (5.25) will be discussed in section 6.2.

The large-H behaviour of the zero-temperature entropy S_0 and of the specific heat amplitude Γ_0 are actually simpler to analyse than the ground-state energy. Indeed, the results concerning S_0 start to be non-trivial when considering the terms of order y_0^ν in the expansion of the polynomials $\tilde{Q}(z)$ and $\tilde{R}(z)$. These terms are correctly predicted by the generic results (5.20) and (5.21). Our last task consists in building the polynomials $Q(z)$ and $R(z)$, with the proper normalizations and conventions of their definitions (4.43, 48), from $\tilde{Q}(z)$ and $\tilde{R}(z)$. To do so, we set

$$Q(z) = a_1\tilde{Q}(z) \qquad R(z) = a_2\tilde{R}(z) + a_3\tilde{Q}(z) \tag{5.26}$$

with unknown y_0-dependent amplitudes a_1, a_2, and a_3. It can be checked that all the requirements about $Q(z)$ and $R(z)$ are met, to leading non-trivial order in y_0, i.e. up to and including terms proportional to y_0^ν, for the choice

$$a_1 = (-1)^\nu(1+\epsilon) \qquad a_2 = (-1)^\nu(1-\epsilon) \qquad a_3 = \frac{(-1)^\nu(1-\epsilon)}{\nu + y_0} \tag{5.27}$$

with the notation $\epsilon = 2^{\nu-1}(-y_0)^\nu/\nu!$. These expressions allow us to evaluate α_1 from equation (5.2), and S_0 from equation (5.1*b*). We thus obtain

$$S_0 = \frac{ps_1}{2}\left[1 - \frac{1}{\nu!}\left(\frac{2J}{H}\right)^\nu + \dots\right]. \tag{5.28}$$

As far as the specific heat is concerned, the leading behaviour of Γ_0 is generated by the terms proportional to $y_0^{\nu-1}$ in the polynomials $Q(z)$ and $R(z)$, which are correctly predicted by the generic perturbative results (5.20) and (5.21). After evaluating α_2 and α_3 according to equation (5.2), we are left with

$$\Gamma_0 = \frac{p}{8J(\nu-1)!}(s_2 + s_1^2 + \pi^2/6)\left(\frac{2J}{H}\right)^\nu + \dots. \tag{5.29}$$

6. The diluted binary model

It has been mentioned already in the introduction that the 'exactly solvable' class of random-field Ising chains described so far contains the symmetric diluted binary model (1.3) as a limiting case, namely where the integer ν goes to infinity, the strength H_{B} of the random fields being normalized according to equation (1.4). In this section, our aim is twofold. We will first analyse, in section 6.1, the low-temperature behaviour of the binary model *per se*, i.e. without reference to the exact solution described in sections 3 to 5. We will then present, in section 6.2, a detailed comparison between both approaches.

6.1. Low-temperature behaviour

The present section is devoted to the low-temperature behaviour of the random-field Ising chain with the diluted binary distribution (1.3), namely

$$R(h) = \frac{p}{2}\left[\delta(h - H_{\mathrm{B}}) + \delta(h + H_{\mathrm{B}})\right] + r\delta(h). \tag{6.1}$$

We consider again the general formalism of section 2. At low temperatures, both the recursion relation (2.4) for the Riccati variables ρ_n and the expression (2.5) for the free energy can be simplified, allowing an explicit determination of the ground-state energy E_0, of the zero-temperature entropy S_0, and of the amplitude of the specific heat, which falls off exponentially in the present case. As far as E_0 and S_0 are concerned, our approach is very close to that used, for example, in [6] in the study of very similar models.

6.1.1. The ground-state energy. We first evaluate the ground-state energy E_0 of the binary model. To do so, it is sufficient to study the asymptotic exponential growth of the Riccati variables ρ_n as $\beta \to \infty$. Along the lines of [6], we set

$$\rho_n \sim \exp(2\beta c_n) \tag{6.2}$$

where it is understood that only the linear growth in β of the exponent is taken into account. It can be derived from equation (2.4) that the new random variables c_n obey the recursion formula

$$\left| \begin{array}{l} c_{n-1} \geqslant J \Rightarrow c_n = h_n + J \\ -J \leqslant c_{n-1} \leqslant J \Rightarrow c_n = h_n + c_{n-1} \, . \\ c_{n-1} \leqslant -J \Rightarrow c_n = h_n - J \end{array} \right. \tag{6.3}$$

When the site label n becomes large, the variables c_n inherit from the ρ_n the crucial property that they admit an asymptotic stationary probability distribution, which is therefore left invariant under the transformation (6.3). Equation (2.5) implies that the ground-state energy can be expressed in terms of an average with respect to this invariant distribution, denoted by $\langle\!\langle \ldots \rangle\!\rangle$, namely

$$E_0 = -J - 2\langle\!\langle \phi_0(c) \rangle\!\rangle \qquad \text{with} \qquad \phi_0(c) = \begin{cases} c - J & \text{if } c > J \\ 0 & \text{if } c \leqslant J \end{cases} \tag{6.4}$$

and where the site label n has been dropped, for the sake of simplicity.

We now proceed to the actual determination of the invariant distribution of the c_n, for the distribution (6.1) of the random magnetic fields h_n. If all the fields h_n are changed into their opposites, so are the c_n. Hence their invariant distribution is even. Moreover, equation (6.3) shows that the difference between c and $\pm J$ has to be a multiple of H_{B}.

We are thus led to look for a discrete invariant distribution of the form

$$c = \pm(J + H_{\mathrm{B}} - mH_{\mathrm{B}}) \qquad \text{with probability } x_m \tag{6.5}$$

where the integer m varies in the range $0 \leqslant m \leqslant N + 1$. Here and throughout the following, we will use the notation

$$2J/H_{\mathrm{B}} = N - 1 + \xi \tag{6.6}$$

with N integer, $N \geqslant 1$, and $0 \leqslant \xi < 1$.

The physical meaning of the integer N will become clearer in the following. Let us just notice that the first value $N = 1$ corresponds to $H_{\mathrm{B}} > 2J$, under which condition the non-zero random fields are strong enough to align the spins along them, independently of their environment.

The probabilities x_m are determined by requiring that the distribution (6.5) be invariant under the action of the transform (6.3). This condition is equivalent to the following coupled linear equations

$$\left| \begin{array}{ll} 2x_0 = p(x_0 + x_{N+1} + x_1) & \\ 2x_1 = 2r(x_0 + x_{N+1} + x_1) + px_2 & \\ 2x_2 = p(x_0 + x_{N+1} + x_1 + x_3) + 2rx_2 & \\ 2x_m = p(x_{m-1} + x_{m+1}) + 2rx_m & (3 \leqslant m \leqslant N - 1) \\ 2x_N = px_{N-1} + 2rx_N & \\ 2x_{N+1} = px_N \, . & \end{array} \right. \tag{6.7}$$

This system can be easily solved, and yields

$$\left|\begin{array}{ll} x_0 = NX & \\ x_1 = \left(\dfrac{2N}{p} - N - 1\right) X & \\ x_m = \dfrac{2}{p}(N+1-m)X & \qquad (2 \leqslant m \leqslant N) \\ x_{N+1} = X. & \end{array}\right. \tag{6.8}$$

The normalization X of the probability weights is fixed by the condition $2\sum_{m=0}^{N+1} x_m = 1$, which yields

$$X = p/[2N(N+1)]. \tag{6.9}$$

According to equation (6.4), the ground-state energy E_0 can be expressed in terms of the x_m as $E_0 = -\left[J + 2H_\mathrm{B}x_0 + 2(NH_\mathrm{B} - 2J)x_{N+1}\right]$. The results (6.8, 9) yield thus the following explicit closed-form formula for E_0

$$E_0 = -J - \frac{2p(NH_\mathrm{B} - J)}{N(N+1)}. \tag{6.10}$$

The ground-state energy is thus a piecewise linear function of the field strength H_B. We have $E_0 = -pH_\mathrm{B} - rJ$ for $N = 1$, i.e. $H_\mathrm{B} \geqslant 2J$, $E_0 = -\frac{2}{3}pH_\mathrm{B} - (1 - \frac{1}{3}p)J$ for $N = 2$, i.e. $J \leqslant H_\mathrm{B} \leqslant 2J$, and so on. In particular, the result for $N = 1$ agrees with what is expected, in a regime where each spin with a non-zero random field is frozen in the direction of its magnetic field. It can be checked that E_0 is a continuous function of H_B, but that it presents cusps, with discontinuous slopes, at the values of H_B such that the integer N jumps, i.e. whenever the ratio $2J/H_\mathrm{B}$ is an integer.

The behaviour of the ground-state energy for a small strength of the random fields, which corresponds to large values of the integer N, deserves some special attention. With notation (6.6), expression (6.10) of E_0 can be expanded for small H_B as

$$E_0 = -J - \frac{pH_\mathrm{B}^2}{2J}\left[1 - \frac{H_\mathrm{B}}{2J} + \frac{H_\mathrm{B}^2}{4J^2}(1 + \xi - \xi^2) + \ldots\right]. \tag{6.11}$$

The term of relative order $(H_\mathrm{B}/J)^2$ involves a function of ξ, meaning that this term exhibits a periodic oscillatory behaviour as a function of J/H_B. We shall return to this aspect in section 6.2, in connection with the exact solution.

6.1.2. The zero-temperature entropy. In order to determine the zero-temperature entropy S_0 of the model, along the lines of [6], we have to make the large-β estimate (6.2) of the Riccati variables more accurate, keeping track of prefactors. We therefore set

$$\rho_n \sim a_n \exp(2\beta c_n). \tag{6.12}$$

Equation (2.4) implies that the couple of random variables (c_n, a_n) obeys the following recursion relation

$$\left|\begin{array}{ll} c_{n-1} > J \Rightarrow c_n = h_n + J & a_n = 1 \\ c_{n-1} = J \Rightarrow c_n = h_n + J & a_n = a_{n-1}/(1+a_{n-1}) \\ -J < c_{n-1} < J \Rightarrow c_n = h_n + c_{n-1} & a_n = a_{n-1} \\ c_{n-1} = -J \Rightarrow c_n = h_n - J & a_n = a_{n-1} + 1 \\ c_{n-1} < -J \Rightarrow c_n = h_n - J & a_n = 1. \end{array}\right. \tag{6.13}$$

The couple (c_n, a_n) is distributed, asymptotically for large n, according to a limit probability distribution, which is invariant under the transformation given in equation (6.13). By expanding expression (2.5) of the free energy for large β in an appropriate way, the zero-temperature entropy S_0 can be expressed as an average with respect to this invariant distribution, namely

$$S_0 = \langle\langle \phi_1(c,a) \rangle\rangle \qquad \text{with} \qquad \phi_1(c,a) = \begin{cases} \ln a & \text{if } c > J \\ \ln(a+1) & \text{if } c = J \\ 0 & \text{if } c < J. \end{cases} \tag{6.14}$$

In analogy with the calculation of the ground-state energy, presented in section 6.1.1, we now proceed to the explicit determination of the stationary distribution of the couple (c, a). The evenness of the distribution of the random fields implies that the values (c, a) and $(-c, 1/a)$ occur with equal weights. By inspection of the recursion (6.13), one realizes that the invariant probability distribution has the following discrete form

$$\{c = \pm(J + H_\mathrm{B} - mH_\mathrm{B}),\ a = k^{\mp 1}\} \qquad \text{with probability } x_{m,k} \tag{6.15}$$

with $0 \leqslant m \leqslant N+1$, as previously, and $k \geqslant 1$ can be an arbitrary integer. The probabilities $x_{m,k}$ are determined by the requirement that the distribution (6.15) be invariant under the action of the transform (6.13). This condition provides the following system of coupled linear equations

$$\left|\begin{array}{l} 2x_{0,1} = p(x_0 + x_{N+1}) \\ 2x_{1,1} = 2r(x_0 + x_{N+1}) + px_{2,1} \\ 2x_{2,1} = p(x_0 + x_{3,1} + x_{N+1}) + 2rx_{2,1} \\ 2x_{0,k} = px_{1,k-1} \qquad (k \geqslant 2) \\ 2x_{1,k} = 2rx_{1,k-1} + px_{2,k} \qquad (k \geqslant 2) \\ 2x_{2,k} = p(x_{1,k-1} + x_{3,k}) + 2rx_{2,k} \qquad (k \geqslant 2) \\ 2x_{m,k} = p(x_{m-1,k} + x_{m+1,k}) + 2rx_{m,k} \qquad (3 \leqslant m \leqslant N-1) \\ 2x_{N,k} = px_{N-1,k} + 2rx_{N,k} \\ 2x_{N+1,k} = px_{N,k} \end{array}\right. \tag{6.16}$$

where the quantities with one single subscript are as in section 6.1.1.

The system (6.16) can also be solved in closed form, and yields

$$\left|\begin{array}{ll} x_{0,k} = NX\ p_N r_N^{k-1} & \\ x_{1,k} = \dfrac{2}{p} NX\ p_N r_N^{k} & \\ x_{m,k} = \dfrac{2}{p}(N+1-m)X\ p_N r_N^{k-1} & \qquad (2 \leqslant m \leqslant N) \\ x_{N+1,k} = X\ p_N r_N^{k-1} & \end{array}\right. \tag{6.17}$$

where the normalization X is still given by equation (6.9), and with the notation

$$p_N = \frac{(N+1)p}{2N} \qquad r_N = 1 - p_N. \tag{6.18}$$

According to equation (6.14), the zero-temperature entropy S_0 can be expressed as

$$S_0 = \sum_{k \geqslant 1} \left[x_{1,k} \ln \frac{k+1}{k} + (x_{N+1,k} - x_{0,k}) \ln k \right]. \tag{6.19}$$

The solution (6.17) leads to an explicit expression for S_0, namely

$$S_0 = \frac{p^2}{2N^2} \sum_{k \geqslant 1} r_N^{k-1} \ln k. \tag{6.20}$$

The zero-temperature entropy depends on the strength H_{B} of the fields only through the integer N. It is therefore a discontinuous function of the field strength H_{B}. Unlike in the 'exactly solvable' model, the dependences of S_0 on p and on H_{B} do not factorize. Finally, S_0 assumes non-trivial limits in the non-diluted case ($p = 1$), for $N \geqslant 2$. We will comment some more about these matters in section 6.2, and in the discussion.

Moreover, when the ratio $2J/H_{\mathrm{B}}$ is exactly an integer, the result (6.20) is not valid. Indeed, there are some extra degeneracies, so that the right-hand sides of the system (6.16) contain some additional terms. These points correspond to discontinuous 'spikes' in the dependency of the zero-temperature entropy on the field strength H_{B}: when $2J/H_{\mathrm{B}}$ is exactly an integer, the value of S_0 is larger than its limits from both sides. Such a phenomenon was already underlined in [6], which contains the results (6.10) and (6.19) in the non-diluted case ($p = 1$).

6.1.3. The specific heat. We now want to investigate the low-temperature behaviour of the specific heat of the diluted binary model. To do so, we have to make the estimate (6.12) of the Riccati variables still more accurate, by keeping track of the exponentially small corrections to the leading behaviour (6.12). We set therefore

$$\rho_n = a_n \mathrm{e}^{2\beta c_n} \left(1 + b_n \mathrm{e}^{-2\beta g_n} + \ldots\right). \tag{6.21}$$

Just as previously, by expanding equation (2.4) in an appropriate way, we can derive a recursion relation for the four random variables (c_n, a_n, g_n, b_n). These formulae are rather lengthy, and will not be needed in the following in their full generality.

We give hereafter the transformation law for the leading correction exponent g_n, which depends only on c_{n-1} and g_{n-1}, according to

$$\left|\begin{array}{l} c_{n-1} > J \Rightarrow g_n = c_{n-1} - J \\ c_{n-1} = J \Rightarrow g_n = \mathrm{Inf}\,\{g_{n-1};\ 2J\} \\ 0 \leqslant c_{n-1} < J \Rightarrow g_n = \mathrm{Inf}\,\{g_{n-1};\ J - c_{n-1}\} \\ -J < c_{n-1} < 0 \Rightarrow g_n = \mathrm{Inf}\,\{g_{n-1};\ J + c_{n-1}\} \\ c_{n-1} = -J \Rightarrow g_n = \mathrm{Inf}\,\{g_{n-1};\ 2J\} \\ c_{n-1} < -J \Rightarrow g_n = -c_{n-1} - J \end{array}\right. \tag{6.22}$$

where the function Inf $\{x;\ y\}$ is defined as the smaller of its arguments.

For n large enough, g_n will assume the smallest positive value which is consistent with the recursion (6.22). It can be checked that this value, denoted in the following by $\tilde{g}$, reads

$$\tilde{g} = \mathrm{Inf}\,\{\tilde{g}_1;\ \tilde{g}_2\}$$

with

$$\tilde{g}_1 = 2J - (N-1)H_{\mathrm{B}} = \frac{2J\xi}{N-1+\xi} \qquad \tilde{g}_2 = NH_{\mathrm{B}} - 2J = \frac{2J(1-\xi)}{N-1+\xi}. \tag{6.23}$$

This result shows explicitly that the position of the parameter ξ with respect to the special value $\xi = \frac{1}{2}$ plays a special role in the problem.

The correction exponent $\tilde{g}$ governs the low-temperature behaviour of the specific heat. It can indeed be shown that the free energy admits the following low-temperature expansion

$$F = E_0 - S_0\,T - \tilde{B}\,T\,\mathrm{e}^{-2\tilde{g}/T} + \ldots \tag{6.24}$$

where the amplitude $\tilde{B}$ can be expressed as some average with respect to the stationary joint distribution of the variables (c_n, a_n, g_n, b_n), restricted to $g_n = \tilde{g}$.

In other words, $2\tilde{g}$ represents the non-vanishing energy gap between the lowest elementary excitations and the ground-states. As a consequence of equation (6.24), the specific heat $C(T)$ has the following exponential fall-off at low temperature

$$C \sim 4\tilde{B}(\tilde{g}/T)^2 \mathrm{e}^{-2\tilde{g}/T}. \tag{6.25}$$

Figure 1 shows a plot of $\tilde{g}$, against the field strength H_{B}, both in units of J. It can be shown from its expression (6.23) that $\tilde{g}$ is a continuous and piecewise linear function of H_{B}, which oscillates infinitely many times between both its extreme behaviours $\tilde{g} = 0$ and $\tilde{g} = H_{\mathrm{B}}/2$, reached respectively when the ratio $2J/H_{\mathrm{B}}$ is integer, or half-integer.

The explicit evaluation of the amplitude $\tilde{B}$ involves lengthy calculations, which we prefer not to reproduce here, since the derivations follow closely those of sections 6.1.1 and 6.1.2, and do not present any qualitatively new kind of difficulty. It turns out that three generic cases have to be dealt with separately. We just give below our final expressions for the amplitude $\tilde{B}$ in each generic case.

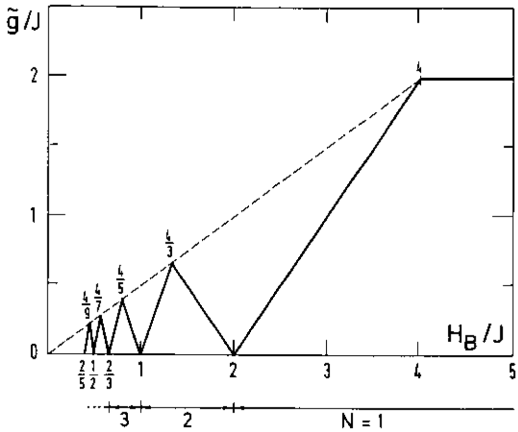

Figure 1. Plot of the reduced exponent $\tilde{g}/J$, against the ratio H_{B}/J, of the low-temperature behaviour (6.25) of the specific heat of the binary model. $\tilde{g}$ oscillates infinitely many times between $\tilde{g} = 0$, when $2J/H_{\mathrm{B}}$ is integer, and $\tilde{g} = H_{\mathrm{B}}/2$ (shown as a dotted line), when $2J/H_{\mathrm{B}}$ is half-integer.

(i) $0 < \xi < \frac{1}{2}$, i.e. $\tilde{g} = \tilde{g}_1$ and $N = 2$

$$\tilde{B} = \frac{6N + 2 + p_N}{3N^2(N+1)^2 p_N}. \tag{6.26}$$

This amplitude diverges for small p as

$$\tilde{B} \approx \frac{4(3N+1)}{3N(N+1)^3\, p}. \tag{6.27}$$

This striking property will be explained in more detail in the appendix, in terms of a non-trivial crossover phenomenon between the non-commuting $p \to 0$ and $T \to 0$ limits.

(ii) $\frac{1}{2} < \xi < 1$, i.e. $\tilde{g} = \tilde{g}_2$

$$\tilde{B} = \frac{p^3}{4N^3}\left(\frac{\ln p_N}{r_N}\right)^2. \tag{6.28}$$

The structure of this expression is very different from (6.26). In particular, in the present case, the amplitude $\tilde{B}$ vanishes rapidly for $p \to 0$, as

$$\tilde{B} \approx \frac{p^3 \ln^2 p}{4N^3}. \tag{6.29}$$

(iii) $H_{\mathrm{B}} > 4J$ ($N = 1$ and $0 < \xi < \frac{1}{2}$)

In this case, the lowest excitations consist in flipping clusters of consecutive spins which feel a zero magnetic field. Such a flip costs an energy $4J$, in agreement with the expression $\tilde{g} = 2J$. The structure of the equations is slightly different from the generic ones. As a consequence, our final expression for the amplitude $\tilde{B}$ is also somehow different, namely

$$\tilde{B} = \frac{(4-p)r}{6p}. \tag{6.30}$$

This last result, which shares with equation (6.26) the property of blowing up as $1/p$ for small p, will be met again, via a different approach, in the appendix, devoted to the $H_{\rm B} = \infty$ limit.

The other case with $N = 1$, namely $\frac{1}{2} < \xi < 1$, i.e. $2J < H_{\rm B} < 4J$, is not different from the generic case (ii).

When the ratio $4J/H_{\rm B}$ is an integer, namely when the variable ξ equals either zero or one half, the situation is again more complex, because of some extra degeneracies. We have not evaluated the amplitude $\tilde{B}$ for these particular cases.

6.2. Connection with the exact solution

In this section, we want to discuss the connection between the exact results obtained in sections 4 and 5, concerning the probability distribution (1.2) of the random magnetic fields, with the results of section 6.1, concerning the binary distribution (1.3). As mentioned already in the introduction, the diluted symmetric binary distribution is nothing but the $\nu \to \infty$ limit of the power-times-exponential distribution (1.2), the parameter $H_{\rm B}$ of equation (1.4) being kept fixed.

It can therefore be suspected that the results of our exact solutions exhibit an interesting crossover behaviour when the integer ν is large. In this section, we will focus our attention on zero-temperature properties, leaving the low-temperature behaviour of the specific heat for the discussion.

We start the large-ν analysis of the results of sections 4 and 5 by an evaluation of the complex numbers ω_a, which have played a crucial part in the exact solution. We recall that these numbers are the $\nu - 1$ roots of equation (4.8), or (4.9), with strictly positive real parts. It turns out that three regimes have to be discussed separately.

(i) For large ν, and $\mathrm{Re}\,\omega > 0$ fixed, $(1+\omega)^{-\nu}$ is exponentially negligible with respect to $(1-\omega)^{-\nu}$. We therefore obtain the estimate

$$\omega_a \approx 1 - 2^{-1/\nu} \mathrm{e}^{-2\pi i a/\nu} \qquad (1 \leqslant a \leqslant \nu - 1). \tag{6.31}$$

This result, which expresses that the roots are equally spaced on the circle centred at the point $\omega = 1$ with radius $2^{-1/\nu} \approx 1$, holds when both integers ν and a are large and comparable, provided their ratio a/ν does not approach the limiting values 0 or 1. These limits, for which the real parts of the roots (6.31) go to zero, have to be dealt with in a different way.

It turns out that the relative positions of the roots ω_a will not be altered with respect to (6.31). We will thus assume throughout the following that the roots ω_a go counter-clockwise around the point $\omega = 1$ when the label a runs from 1 to $\nu - 1$, and that $\omega_{\nu-a}$ is the complex conjugate of ω_a. In particular, if ν is an even integer, there exists one real root, namely $\omega_{\nu/2}$.

(ii) For large ν, keeping a fixed, we look for a solution of the form $\omega_a \approx Z/\nu$. Equation (4.8) can then be expanded as

$$\cosh Z + \frac{Z^2}{2\nu} \sinh Z + \frac{Z^3}{3\nu^2} \cosh Z + \ldots = 1. \tag{6.32}$$

Keeping only the leading order, we obtain $Z = 2\pi i a$, where a is an arbitrary integer. The higher-order terms can be taken into account in a systematic way. We thus obtain the following large-ν behaviour of the first roots ω_a $(1 \leqslant a \ll \nu)$

$$\omega_a = \frac{2\pi i a}{\nu}\left[1 - \frac{1}{\nu} + \frac{1}{\nu^2} + \ldots\right] + \frac{2\pi a}{\nu^{3/2}}\left[1 + \frac{1}{\nu}\left(\frac{\pi^2 a^2}{3} - 1\right) + \ldots\right] \tag{6.33}$$

(iii) This last regime consists in the non-trivial crossover between (i) and (ii) which takes place for values of the index a such that a^2 and ν are comparable. The occurrence of such a crossover phenomenon is suggested by the form of the first correction term to the real part of ω_a in equation (6.33). We set, for the sake of convenience

$$X = \frac{2\pi a}{\nu^{1/2}} \qquad \omega_a = \frac{2\pi i a + Y}{\nu} \tag{6.34}$$

where it is understood that a and ν go to infinity, X and Y remaining finite. In this regime, equation (4.8) assumes the simpler form $\cosh Y = \exp(X^2/2)$, whence

$$Y = \frac{X^2}{2} + \ln\left(1 + \sqrt{1 - \exp(-X^2)}\right). \tag{6.35}$$

Notice that the Y-variable is real: only the real part of the roots ω_a is affected by the effect under consideration, to leading order in $\nu^{1/2}$. When the scaling variable X is small, this solution behaves as $Y \approx X + X^3/12$, in accord with the expansion (6.33) of $\mathrm{Re}\,\omega_a$. In the converse large-X limit, we have $Y \approx X^2/2 + \ln 2$, in agreement with equation (6.31), up to exponentially small corrections.

We end up this discussion with a numerical illustration. Figure 2 shows a plot of the roots ω_a, and their opposites, in the complex ω-plane, for $\nu = 10$ and $\nu = 25$. These plots are very close to being the circles predicted by equation (6.31). Indeed, crossover behaviour (iii) only sets in for very large values of ν. This can be realized by noticing that the first root $(a = 1)$ is fully in the crossover regime $(X \approx 1)$ only for $\nu \sim 40$. Unfortunately, this very late convergence towards the true asymptotic large-ν limit will be shared by most physical quantities.

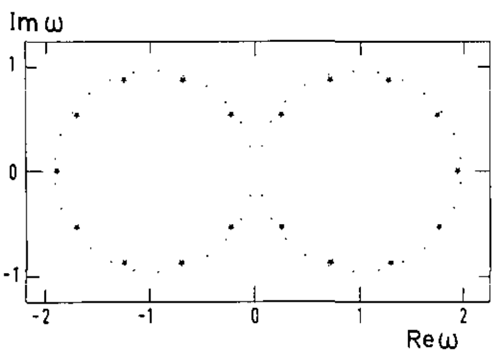

Figure 2. Plot of the roots $\pm\omega_a$ $(1 \leqslant a \leqslant \nu - 1)$ of the secular equation (4.9), in the complex ω-plane, for $\nu = 10$ (stars), and $\nu = 25$ (dots).

We are now able to go back to the small-H limit, considered in section 5.3. Our first aim consists in evaluating the sums $\Omega_{(1)}$ and $\Omega_{(-1)}$, defined in equation (5.13), which determine, via equation (5.12), the $H \to 0$ limit of all the low-temperature thermodynamical properties. To leading order for large ν, these sums can be evaluated by means of equation (6.31). We thus obtain

$$\Omega_{(1)} \approx \nu \qquad \Omega_{(-1)} \approx \nu/2. \tag{6.36}$$

The first correction to these asymptotic results originates in the crossover regime (iii) described earlier, i.e. in values of a, or of $(\nu - a)$, of order $\nu^{1/2}$. By inserting the result (6.34) and (6.35) into the definition (5.13), subtracting the circular law (6.31) in an appropriate way, and finally converting the sums over the integer a into integrals over the continuous variable X, we obtain the following estimates

$$\Omega_{(1)} \approx \nu - \frac{A_{(1)}}{\nu^{1/2}} \qquad \Omega_{(-1)} \approx \frac{\nu}{2} + \frac{\nu^{1/2}}{2\pi} \ln \frac{\nu}{\nu_0} \tag{6.37}$$

with

$$A_{(1)} = -\frac{1}{\pi} \int_0^{+\infty} \mathrm{d}X \left[\ln \left(1 + \sqrt{1 - \exp(-X^2)} \right) - \ln 2 \right] \approx 0.119\,899\,86$$

$$\nu_0 = 4\pi^2 \exp \left\{ -2 \int_0^{+\infty} \frac{\mathrm{d}X}{X^2} \left[\ln \left(1 + \sqrt{1 - \exp(-X^2)} \right) - X\mathrm{e}^{-X} \right] \right\} \approx 8.166\,752\,75. \tag{6.38}$$

By inserting estimates (6.37) and (6.38) into equations (5.12) and (5.1), we are able to study how the small-H_{B} behaviour of the thermodynamical quantities approaches, for large ν, the results concerning the binary distribution, derived in section 6.1.

Considering first the ground-state energy, the first explicit ν-dependence shows up in the H_{B}^3 terms, namely

$$E_{0(\text{binary})} - E_0 \approx \frac{pH_{\mathrm{B}}^3}{4\pi J^2} \left(\frac{1}{\nu^{1/2}} \ln \frac{\nu}{\nu_0} \right) + \dots. \tag{6.39}$$

The expression inside the parentheses exhibits a broad maximum around $\nu = \mathrm{e}^2 \nu_0 \approx 60$, where it equals 0.0821, before it falls off very slowly to zero.

As far as the specific heat amplitude Γ_0 is concerned, the estimate (6.37) implies the following behaviour, for H_{B} small, and ν large

$$\Gamma_0 \approx \frac{pA_{(1)} H_{\mathrm{B}}}{4J^2} \nu^{1/2} \left(s_2 + s_1^2 + \pi^2/6 \right). \tag{6.40}$$

We will come back to this growth in $\nu^{1/2}$ in the discussion.

We now want to address the more difficult question of the convergence of the low-temperature thermodynamical quantities, and especially of the ground-state energy E_0, for large ν, towards their expressions in the binary limit, for arbitrary values of the scaled field strength H_{B}. This study will in particular shed some light on the range of validity of the estimates (6.39, 40).

Let us start by recasting the main result (6.10) concerning the binary model, within the formalism of the exact solution in the $\nu \to \infty$ limit. To do so, we evaluate first the function $C(y)$, defined in section 4.1 as the scaled low-temperature limit, under the change of variable (4.1), of the moments C_k, defined in equations (3.17) and (3.18). Using the definitions (2.10), (3.18), (4.1) and (6.2), we obtain the estimate

$$C(y) \approx \langle\!\langle \exp\left[-\mathrm{e}^{2\beta(Hy+J-|c_n|)}\right]\rangle\!\rangle \qquad (y > 0, \nu \to \infty) \tag{6.41}$$

where $\langle\!\langle \ldots \rangle\!\rangle$ denotes an average over the invariant distribution of the variables c_n, studied in section 6.1. It is then natural to scale the variable y according to $y = \nu Y$, and to rewrite equation (6.41) as

$$C(\nu Y) \approx \langle\!\langle \theta\left(|c_n| - J - H_\mathrm{B} Y\right)\rangle\!\rangle \qquad (Y > 0). \tag{6.42}$$

The right-hand side of this last expression can be evaluated in closed form, using the invariant distribution (6.5). It turns out that there are only two relevant values of the random variable $|c_n|$, namely $|c_n| = J + H_\mathrm{B}$ and $|c_n| = NH_\mathrm{B} - J$, with respective probabilities $2x_0$ and $2x_{N+1}$, given by equations (6.8) and (6.9). We are thus left with the estimate

$$C(\nu Y) \approx \frac{1}{N(N+1)}\left[N\ \theta(1-Y) + \theta(1-\xi-Y)\right] \tag{6.43}$$

where N and ξ have been defined in equation (6.6). Finally, using equations (4.27) and (4.42), we can derive from the estimate (6.43) the following large-ν limit of the polynomial $Q(z)$, up to an irrelevant normalization factor, and with the variable rescaling $z = Z/\nu$

$$Q(Z/\nu) \approx \mathcal{Q}(Z) = \frac{1}{Z}\left[N\left(1-\mathrm{e}^{-Z}\right) + \mathrm{e}^{-\xi Z} - \mathrm{e}^{-Z}\right] \qquad (\nu \to \infty). \tag{6.44}$$

This expression is indeed formally a polynomial of infinite degree, i.e. an entire function.

We can now estimate from the result (6.44) the large-ν behaviour of the parameter α_0 as

$$\frac{\alpha_0}{\nu} \approx 1 + \frac{\mathcal{Q}'(0)}{\mathcal{Q}(0)} = \frac{N+1-2\xi+\xi^2}{2(N+1-\xi)} \tag{6.45}$$

and check that the resulting expression for the ground-state energy, obtained by inserting the result (6.45) into equation (5.1a), coincides with equation (6.10), as it should.

On the other hand, for large ν and in terms of the scaled form $\mathcal{Q}(Z)$, the condition (4.44) to be fulfilled by the polynomial $Q(z)$ only involves the form (ii) of the roots ω_a, described earlier in this section, and given in equation (6.33). To leading order for large ν, we obtain the condition

$$\mathcal{Q}(2\pi \mathrm{i} a) = \mathrm{e}^{2\pi \mathrm{i} a \xi}\mathcal{Q}(-2\pi \mathrm{i} a) \qquad (a \geqslant 1). \tag{6.46}$$

It is easy to check that the result (6.44) indeed obeys this identity. Surprisingly enough, the condition (6.46) does not 'feel' at all the dependency of the function $\mathcal{Q}(Z)$ given in equation (6.44) on the essential integer parameter N. In other words, $Q(Z)$ is just one among the infinity of entire functions which obey equation (6.46) ; this special solution is not selected by the condition (6.46) alone, albeit by more global features of the problem.

A natural way of dealing with the large-ν behaviour of the exact solution, and of comparing its outcomes with those of the binary model, would consist in performing a perturbative analysis, and to look for an expansion of the polynomial $Q(z)$ around the limit expression (6.44), which would hold for large ν. We have not been able to achieve this programme, seemingly because of the difficulties alluded to in the previous paragraph.

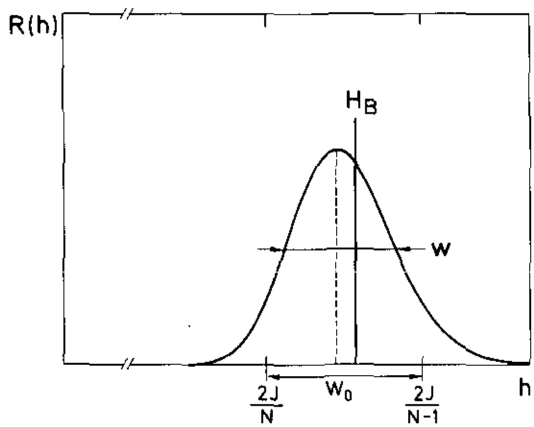

Figure 3. Schematic plot of the continuous part of the probability density (1.2), for ν large, and magnetic fields h close to their average absolute value $H_{\mathrm{B}} = \nu H$, illustrating the role of the two relevant energy scales w and W_0, defined in the text.

We will, instead, present a heuristic approach, which provides a qualitative understanding of the corrections to the binary limit (6.10) of the ground-state energy. The essence of the argument is shown in figure 3. When the integer ν is large, the probability distribution of the magnetic fields h_n is sharply peaked around the value $H_{\mathrm{B}} = \nu H$. From a more quantitative viewpoint, the (root-mean-square) width w of the continuous part of the probability distribution (1.2) reads

$$w^2 = \langle h^2 \rangle - \langle |h| \rangle^2 = \nu H^2 = H_{\mathrm{B}}^2/\nu. \tag{6.47}$$

It is clear from figure 3 that the relevant energy scale, to which the width w of the disorder is to be compared, is given by the width W_0 of the interval determined by the adjacent special values $2J/(N-1)$ and $2J/N$ of the field strength H_{B}, which have played a central part in the study of the binary model. We have thus $W_0 = 2J/[N(N-1)] \approx 2J/N^2 \approx H_{\mathrm{B}}^2/(2J)$ in the small-field limit. We are thus led to consider successively the following two regimes: (i) $\nu \ll N^2$ and (ii) $\nu \gg N^2$.

(i) $\nu \ll N^2$, i.e. $w \gg W_0$: This first regime sets in when the field strength H_{B} is very small, at fixed ν, namely $H_{\mathrm{B}} \ll \nu^{-1/2} J$. It thus encompasses the usual small-H limit, which has been studied in section 5.3. Conversely, the field strength H_{B} being fixed, regime (i) corresponds to small values of ν, for which the system is very sensitive to the difference between the power-times-exponential distribution, and the binary one. Finally, (i) is paradoxically a large-disorder regime, since the width of disorder w is very large, compared with the interval width W_0.

The result (5.16) and (5.17) shows that the leading non-trivial term in the small-H behaviour of the ground-state energy is proportional to

$$E_0 - E_{0(\mathrm{binary})} \sim \mathrm{Re}\ t_1 \sim \mathrm{Re}\ \exp\left(-\frac{2J}{H}\omega_1\right) \sim \cos\left(\frac{4\pi J}{H_\mathrm{B}}\right) \exp\left[\frac{-4\pi J}{(H_\mathrm{B}\nu^{1/2})}\right] \qquad (6.48)$$

where the rightmost estimate has been obtained using the expansion (6.33) of the real and imaginary parts of the first root ω_1. Roughly speaking, the contribution of the next root ω_2 scales as the square of the first one, and so on.

The argument of the exponential function in the rightmost side of equation (6.48) can be recast as $2\pi N/\nu^{1/2}$. As a consequence, the condition $\nu \ll N^2$ which defines regime (i) can also be thought of as the condition of validity of the small-field results derived in section 5.3, and of the estimates (6.39) and (6.40), which were based on those results.

(ii) $\nu \gg N^2$, i.e. $w \ll W_0$: This second regime corresponds to large values of the integer parameter ν, at fixed field strength H_B, so that the width w is very small, and that the power-times-exponential distribution (1.2) becomes hardly distinguishable from the symmetric binary one (1.3). As far as the ground-state energy is concerned, two cases have to be discussed separately, namely $N = 1$ and $N \geqslant 2$.

For $N = 1$, i.e. $H_\mathrm{B} > 2J$, an overwhelming majority of the spins feel a random field such that $|h_n| > 2J$. These spins are therefore aligned with their random fields, and their contribution to E_0 depends on the distribution of the fields only through $\langle |h| \rangle = H_\mathrm{B}$, independently of ν. As a consequence, the difference between the ground-state energy and its value in the binary case is entirely due to the fraction of random fields such that $|h| \leqslant 2J$. This concentration of unpinned spins can be evaluated for large ν by expanding the probability density (1.2). We thus obtain

$$E_0 - E_{0(\mathrm{binary})} \sim \exp\left[\nu(1 - \xi + \ln\xi)\right] \qquad (N = 1) \qquad (6.49)$$

with $\xi = 2J/H_\mathrm{B}$, according to the definition (6.6). The correction to the ground-state energy is therefore exponentially small in ν. When the field strength H_B is large, ξ is small, and the correction assumes the form $(\xi e)^\nu$, in agreement with the large-H result (5.25), after rescaling the field strength H according to equation (1.4).

In the converse situation where $N \geqslant 2$, i.e. $H_\mathrm{B} < 2J$, the ground-states of the binary model cannot be simply described in terms of spins pinned by their local fields. This can be realized e.g. by noticing that the zero-point entropy S_0, given by equation (6.20), remains non-trivial in the non-diluted case ($p = 1$). When ν is large, but finite, the degeneracy among the ground-states is partially lifted, yielding a correction to E_0 of order the width w of the distribution, given by equation (6.47). This argument leads us to conjecture the following expression

$$E_0 - E_{0(\mathrm{binary})} \sim \frac{pJ\,\mathcal{A}}{\nu^{1/2}} \qquad (N \geqslant 2) \qquad (6.50)$$

where the dimensionless amplitude $\mathcal{A}$ can only depend on the ratio H_B/J. As mentioned earlier, we have not succeeded in deriving this dependency from the formalism of the exact solution. It seems likely that the amplitude $\mathcal{A}$ is discontinuous at integer values of the ratio $2J/H_\mathrm{B}$, in analogy with the zero-point entropy of the binary model.

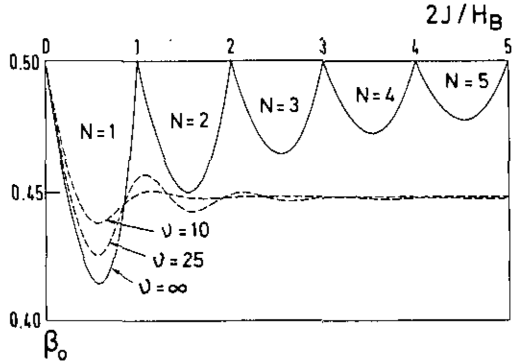

Figure 4. Plot of the reduced parameter $\beta_0 = \alpha_0/(\nu+1)$, which enters the expression of the ground-state energy E_0 of the class of 'exactly solvable' models, against the ratio $2J/H_B$: full line, exact expression (6.52) in the $\nu \to \infty$ limit, extracted from the ground-state energy (6.10) of the binary model; dotted lines, outcome of the 'exact solution', for $\nu = 10$ and $\nu = 25$, determined by solving the linear system (5.3) numerically.

We end up this lengthy section by the following numerical illustration. Figure 4 shows a plot of the scaled ground-state energy parameter

$$\beta_0 = \frac{\alpha_0}{\nu + 1} \tag{6.51}$$

against the ratio $2J/H_B$, in the binary model ($\nu = \infty$, full line), and for two values ($\nu = 10$ and 25) of the exactly solvable model. The normalization (6.51) is chosen so that the intercept reads $\beta_0 = 1/2$ for $H \to \infty$, independently of ν, as can be checked from the results of section 5.4. The result concerning the binary model has been obtained by comparing equations (5.1a) and (6.10), namely

$$\beta_0 = \frac{1}{2} - \frac{\xi(1-\xi)}{2(N+1-\xi)}. \tag{6.52}$$

This function exhibits an infinity of damped periodic oscillations below the value $\frac{1}{2}$. The results for finite ν, obtained by solving numerically the linear system (5.3), exhibit a slow convergence to the limit (6.51). We notice, in particular, to the right of the figure, the small-field limits, which can be evaluated from the result (5.12), which numerically yields $\beta_0 \approx 0.447\,649$ for $\nu = 10$, and $\beta_0 \approx 0.447\,075$ for $\nu = 25$. The form of the first correction to these limits is given by equation (6.48): it has the observed exponentially damped oscillatory behaviour. Finally, the number of appreciably visible oscillations can be estimated to be of order $N \sim \nu^{1/2}$, from the crossover between regimes (i) and (ii), discussed earlier in this section.

7. Discussion

We have presented an 'exact solution' of the random-field Ising chain, where the magnetic fields have the *diluted symmetric power-times-exponential* form (1.2), with an arbitrary positive integer exponent $(\nu - 1)$. We have solved the model at any finite temperature (section 3), in the sense that the integration over the distribution of the random fields has been performed in an exact way. The solution of the problem takes the form of coupled linear recursion relations, which can yield numerical values of thermodynamical quantities, at any finite temperature, with essentially arbitrary accuracy.

We have then analysed in more detail the low-temperature regime (sections 4 and 5), obtaining in particular exact expressions for the ground-state energy E_0, the zero-point entropy S_0, and the amplitude Γ_0 of the specific heat, which exhibits a linear law of the form $C(T) \approx \Gamma_0 T$ at low temperature. The whole non-triviality of these outcomes consists in the presence of four dimensionless parameters, denoted by α_0, α_1, α_2 and α_3.

The results of the present analysis generalize to arbitrary values of the integer parameter ν those obtained in our previous works ([1–3]), which concerned only the cases $\nu = 1$ and $\nu = 2$. With respect to these previous studies, the main complication which is met by increasing the integer ν consists in the occurrence of the $(\nu-1)$ complex roots ω_a of the secular equation (4.9), which have prevented us from obtaining our final results in fully closed form.

As a matter of fact, the occurrence of these complex roots can be related to earlier works. Derrida and Hilhorst [8] have considered the random-field Ising chain, with a generic non-symmetric distribution of the random fields h_n, in the limit where the ferromagnetic coupling J is much larger than the typical random fields. In this regime, under the hypothesis $\langle h\rangle > 0$, these authors predict that the quenched free energy has generically an exponentially small singular part, of the form

$$F \approx -J - \langle h\rangle - C\mathrm{e}^{-4\alpha^\star J} + \cdots \tag{7.1}$$

where $\alpha^\star$ is the (temperature-dependent) real positive root of the equation

$$f(\alpha) = 1 \tag{7.2}$$

with the definition

$$f(\alpha) = \langle\exp(-2\alpha h_n)\rangle. \tag{7.3}$$

$\alpha^\star$ depends thus in a continuous way on the possible parameters of the distribution of the random fields. An analogous continuously varying exponent also shows up in the study of the probability distribution of the variable

$$z = 1 + x_1 + x_1 x_2 + x_1 x_2 x_3 + \cdots \tag{7.4}$$

where the x_n are independent random variables, with a common given distribution (see [19], and references therein).

In the case of the 'exactly solvable' power-times-exponential distribution (1.2), the function $f(\alpha)$ assumes the following form

$$f(\alpha) = r + \frac{p}{2}\left[(1+2\alpha H)^{-\nu} + (1-2\alpha H)^{-\nu}\right] \tag{7.5}$$

so that the non-vanishing roots α_a of the equation $f(\alpha) = 1$ are such that $2\alpha_a H = \pm\omega_a$. As a consequence, the singular part of equation (7.1) is of a similar nature as each of the terms $\exp(-2\omega_a J/H)$ which enter our exact results.

One of the most appealing features of the class of 'exactly solvable' models studied here is the large-ν crossover to the diluted symmetric binary distribution. We have studied in detail the low-temperature thermodynamics of this discrete model (section 6.1), obtaining exact closed-form expressions for the ground-state energy, the zero-temperature entropy, and the amplitude of the specific heat, which falls off exponentially at low temperature.

When the integer parameter ν goes to infinity, the probability distribution of the random fields crosses over from continuous to discrete. This change in the nature of the random fields induces interesting crossover phenomena in the low-temperature thermodynamics. We have studied this question in detail at zero temperature (section 6.2), considering the ground-state energy of the model. We have shown how this crossover was dominated by the interplay between two scales of magnetic fields, namely the width $w = H_{\mathrm{B}}/\nu^{1/2}$ of the continuous part of the distribution of the random fields, and the width $W_0 \approx H_{\mathrm{B}}^2/(2J)$ of the interval of field strengths over which the zero-temperature entropy of the binary model is a constant.

It turns out that similar crossover phenomena do persist at finite temperature. In the $\nu \to \infty$ limit, the free energy of the 'exactly solvable' model goes continuously to that of the diluted binary model. The same property holds true for the ground-state energy E_0, but *not* for the zero-temperature entropy S_0. This quantity is indeed generically larger in the binary model that in the $\nu \to \infty$ limit of the continuous one. In particular, S_0 does not vanish in the non-diluted case ($p = 1$) of the binary model, for $N \geqslant 2$. The degeneracies which are responsible for this discontinuous behaviour of the entropy at zero temperature will also manifest in the low-temperature specific heat, for temperatures of order $T^\star \sim w = H_{\mathrm{B}}/\nu^{1/2}$.

Finally, it can be speculated that the crossover between the linear law of the specific heat, characteristic of a continuous distribution of random fields, and the exponential law, characteristic of a discrete one, will indeed take place for temperatures of order $T^\star$, and that the specific heat at the crossover temperature is comparable to the difference in zero-temperature entropies between both models, and thus independent of the integer ν. This argument is confirmed, at least in the limit of a small field strength H_{B}, by the $\nu^{1/2}$ law (6.40) derived for the specific heat amplitude Γ_0.

It would, of course, be most desirable to obtain some more quantitative information about the crossover phenomena to which we have just alluded, concerning both the zero- and finite-temperature thermodynamics, within the framework of the exact solution, although these matters seem to be appallingly complicated.

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Appendix. The $H = \infty$ limit

In this appendix, we study the class of 'exactly solvable' random-field Ising chains, with the distribution (1.2) of the magnetic fields, in the limit where the field strength

H is infinite. The random magnetic fields are thus distributed as follows

$$h_n = \begin{cases} +\infty & \text{with probability } p/2 \\ 0 & \text{with probability } r \\ -\infty & \text{with probability } p/2. \end{cases} \tag{A.1}$$

Any dependence in the integer parameter ν has disappeared, since every non-zero random field is infinitely large. Hence the distribution (A.1) is also the $H_{\mathrm{B}} = \infty$ limit of the diluted symmetric binary distribution (1.3).

This limiting case has already been studied in detail by Grinstein and Mukamel [9]. The infinitely large fields have the effect of cutting the chain into an infinity of independent finite clusters, so that the thermodynamics of the problem at finite temperature can be dealt with by an exact enumeration procedure.

For the sake of consistency with the body of this article, let us evaluate the free energy of the model (A.1) by means of the Riccati variables. To do so, we consider the variables Z_n defined in equation (2.10), which obey the recursion relation (2.11). The quenched free energy is then given by the expression (2.13).

Within the present formalism, the peculiarity of the distribution (A.1) manifests itself by a drastic simplification of the recursion (2.11), namely

$$\left| \begin{array}{l} h_n = +\infty \Rightarrow Z_n = -1 \\ h_n = 0 \Rightarrow Z_n = \mathrm{e}^{-2\mu} Z_{n-1} \\ h_n = -\infty \Rightarrow Z_n = 1 \end{array} \right. \tag{A.2}$$

which leads us easily to the following stationary distribution of the variable Z

$$Z = \pm \mathrm{e}^{-2n\mu} \qquad \text{with probability } \frac{p}{2} r^n \qquad (n \geqslant 0) \tag{A.3}$$

where the parameter μ has been defined in equation (2.6).

In order to define properly the free energy of the model (A.1), and along the lines of [9], one has to subtract the (temperature-independent) contribution of the infinite magnetic fields. In other words, one considers the subtracted free energy $F' = \lim_{H_{\mathrm{B}} \to \infty} (F - pH_{\mathrm{B}})$, where F denotes the free energy of the diluted symmetric binary model (1.3). It turns out that the term to be subtracted coincides with the divergent contribution of the value $Z = -1$ to the expression (2.13) of the free energy. We thus obtain for the subtracted free energy F'

$$\beta F' = -r \ln 2 + \tfrac{1}{2} \ln(1 - \mathrm{e}^{-4\mu}) - \frac{p^2}{2} \sum_{n \geqslant 1} r^{n-1} \ln(1 - \mathrm{e}^{-4n\mu}). \tag{A.4}$$

Let us now turn to the low-temperature analysis of this free energy. To do so, we expand first the generic term of the sum in equation (A.4) as

$$\ln(1 - \mathrm{e}^{-4n\mu}) = -2\beta J + \ln(4n) - 2n\mathrm{e}^{-2\beta J} + \tfrac{1}{3}(2n^2 + 1)\mathrm{e}^{-4\beta J} + \cdots. \tag{A.5}$$

By inserting this expansion into equation (A.4), we obtain the following low-temperature behaviour of the free energy

$$\beta F' = \beta E_0' - S_0 - \tilde{B} \mathrm{e}^{-4\beta J} + \cdots \tag{A.6}$$

with

$$E_0' = -rJ \qquad S_0 = \frac{p^2}{2}\sum_{n\geqslant 1} r^{n-1}\ln n = \frac{p\,s_1}{2}$$

$$\tilde{B} = -\frac{1}{2} + \frac{p^2}{6}\sum_{n\geqslant 1} r^{n-1}(2n^2+1) = \frac{(4-p)r}{6p}. \tag{A.7}$$

These outcomes agree both with the low-temperature analysis of our exact solution (sections 4 and 5), and with the results concerning the general binary model (section 6). In particular, the subtracted ground-state energy E_0' coincides with the H-independent terms in both results (5.25), and (6.10) for $N = 1$. The value of the zero-temperature entropy S_0 agrees with the large-H limit of equation (5.28), and with (6.20) for $N = 1$. As far as the specific heat is concerned, in the language of section 6, we have $\tilde{g} = 2J$, and the value of the amplitude $\tilde{B}$ is in agreement with equation (6.30).

The limit distribution (A.1) also allows for a detailed study of the striking phenomenon, already underlined in section 6.1.3, that the specific heat amplitudes $\tilde{B}$ given in equations (6.26) and (6.30) blow up in the small-p limit. In other words, the $p \to 0$ and $T \to 0$ limits do not commute in the diluted binary model. As a matter of fact, the presence of powers of $\ln p$ in the small-p behaviour (4.40) of the quantities s_k suggest that these limits do not commute either in the case of continuous distributions of the random fields.

In heuristic terms, this phenomenon can be understood as follows. At low temperatures, the correlation length $\xi_T = 1/(2\mu)$ of the pure ferromagnetic Ising chain is divergent. When the impurity concentration p is small, there is a second diverging length in the problem, namely the mean distance between two impurities, which reads $\xi_p \sim 1/p$. It can therefore be expected that, when both p and T go to zero, there is a continuum limit, where the two lengths have to be compared, and where physical quantities keep a non-trivial dependence in the ratio of both diverging lengths, that we choose to write in the form

$$X = \frac{2\mu}{p} \sim \frac{\xi_p}{\xi_T}. \tag{A.8}$$

The occurrence of such a scaling behaviour can be checked explicitly in the case of the free energy. Indeed, the expression (A.4) can be recast in an exact fashion as

$$\beta F' = \beta E_0' - S_0 + \beta F_s$$

with

$$\beta F_s = \frac{1}{2}\ln\frac{\sinh(2\mu)}{2\mu} - \frac{p^2}{2}\sum_{n\geqslant 1} r^{n-1}\ln\frac{\sinh(2n\mu)}{2n\mu}. \tag{A.9}$$

In the scaling limit, where both p and μ go to zero, with a fixed value of the variable X, the sum in equation $(A.9)$ can be turned into an integral over the variable $t = np$, whereas the first term is negligible. We are left with the estimate

$$\beta F_s \approx -\frac{p}{2}\mathcal{F}(X) \tag{A.10}$$

where the scaling function $\mathcal{F}$ reads

$$\mathcal{F}(X) = \int_0^\infty \mathrm{d}t\ \mathrm{e}^{-t} \ln \frac{\sinh(tX)}{tX}. \tag{A.11}$$

When its argument X is small, or large, the scaling function has the following asymptotic behaviour

$$\begin{aligned} \mathcal{F}(X) &= \frac{1}{3}X^2 - \frac{2}{15}X^4 + \cdots \qquad (X \to 0) \\ \mathcal{F}(X) &= X - \ln(2X) + \gamma_{\mathrm{E}} + \cdots \qquad (X \to \infty). \end{aligned} \tag{A.12}$$

Notice that the X^2 (respectively X) behaviour at small X (respectively large X) corresponds to the value $\tilde{g} = 2J$ (respectively $\tilde{g} = J$) of the energy gap of the Ising chain in the presence (respectively in the absence) of strong random fields. The scaling law (A.10) describes the full non-trivial crossover between those two limiting situations.

It turns out that the same function $\mathcal{F}(X)$ has been shown to play a role in the scaling analysis of two other disordered systems, namely the study of the electrical conductivity of infinite ladder networks [20], and the two-dimensional Ising model with layered randomness [21, 22]. It has been shown in particular that the function $\mathcal{F}(X)$ has an interesting analytic structure, which has been put in perspective with the Lifshitz mechanism. In particular, $\mathcal{F}(X)$ is indefinitely differentiable but non-analytic at $X = 0$. Moreover, the function $\mathcal{F}(X)$ can be related to the Bernouilli numbers B_n and to the ψ-function, the logarithmic derivative of Euler's Γ-function. With the notations of [20], we have

$$\mathcal{F}(X) = \mathcal{B}(2X) = \sum_{m\geqslant 1} \frac{B_{2m}}{2m}(2X)^{2m} = -\psi\left(\frac{1}{2X}\right) - X - \ln(2X). \tag{A.13}$$

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